

Cannon-Thurston Maps for Surface Groups I: Amalgamation Geometry and Split Geometry

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Abstract

We introduce the notion of manifolds of amalgamation geometry and its generalisation, split geometry. We show that the limit set of any surface group of split geometry is locally connected, by constructing a natural Cannon-Thurston map.

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1 Introduction

1.1 Statement of Results

In this paper and its successor [31], we continue our study of Cannon-Thurston maps and limit sets of Kleinian groups initiated in [28], [29] and [30]. Several questions and conjectures have been made in this context by different authors:

•1 In Section 6 of [10], Cannon and Thurston raise the following problem:

Question: Suppose a closed surface group $\pi_1(S)$ acts freely and properly discontinuously on \mathbb{H}^3 by isometries. Does the inclusion $\tilde{i} : \tilde{S} \rightarrow \mathbb{H}^3$ extend continuously to the boundary?

The authors of [10] point out that for a simply degenerate group, this is equivalent to asking if the limit set is locally connected.

•2 In [21], McMullen makes the following more general conjecture:

Conjecture: For any hyperbolic 3-manifold N with finitely generated fundamental group, there exists a continuous, $\pi_1(N)$ -equivariant map

$$F : \partial\pi_1(N) \rightarrow \Lambda \subset S_\infty^2$$

where the boundary $\partial\pi_1(N)$ is constructed by scaling the metric on the Cayley graph of $\pi_1(N)$ by the conformal factor of $d(e, x)^{-2}$, then taking the metric completion. (cf. Floyd [13])

•3 The author raised the following question in his thesis [26] (see also [1]):

Question: Let G be a hyperbolic group in the sense of Gromov acting freely and properly discontinuously by isometries on a hyperbolic metric space X . Does the inclusion of the Cayley graph $i : \Gamma_G \rightarrow X$ extend continuously to the (Gromov) compactifications?

A similar question may be asked for relatively hyperbolic groups (in the sense of Gromov [15] and Farb [12]).

The question for relatively hyperbolic groups unifies all the above questions and conjectures.

In this paper we introduce the notion of what we call *amalgamation geometry* which is, in a way, a considerable generalisation of the notion of *i-bounded geometry* introduced in [30]. We then generalise it by weakening the hypothesis to the notion of *split geometry*. A crucial step in this paper is to prove:

Theorems 9.2 and 9.3: Let $\rho : \pi_1(S) \rightarrow PSL_2(C)$ be a faithful representation of a surface group with or without punctures, and without accidental

parabolics. Let $M = \mathbb{H}^3/\rho(\pi_1(S))$ be of split geometry. Let i be an embedding of S in M that induces a homotopy equivalence. Then the embedding $\tilde{i} : \tilde{S} \rightarrow \tilde{M} = \mathbb{H}^3$ extends continuously to a map $\hat{i} : \mathbb{D}^2 \rightarrow \mathbb{D}^3$. Further, the limit set of $\rho(\pi_1(S))$ is locally connected.

In fact our methods prove the following considerably stronger result by combining the techniques of this paper with those of [28] and [29]. This is a partial affirmation of McMullen's conjecture above.

Theorem 10.1 : Suppose that $N^h \in H(M, P)$ is a hyperbolic structure of *split geometry* on a pared manifold (M, P) with incompressible boundary $\partial_0 M$. Let M_{gf} denote a geometrically finite hyperbolic structure adapted to (M, P) . Then the map $i : \widetilde{M_{gf}} \rightarrow \widetilde{N^h}$ extends continuously to the boundary $\hat{i} : \widehat{\widetilde{M_{gf}}} \rightarrow \widehat{\widetilde{N^h}}$. If Λ denotes the limit set of \widetilde{M} , then Λ is locally connected.

In [31], we shall show that the Minsky model is of split geometry. Combining this with Theorems 9.2 and 9.3, we shall get

Theorem [31]: Let ρ be a representation of a surface group H (corresponding to the surface S) into $PSl_2(C)$ without accidental parabolics. Let M denote the (convex core of) $\mathbb{H}^3/\rho(H)$. Further suppose that $i : S \rightarrow M$, taking parabolic to parabolics, induces a homotopy equivalence. Then the inclusion $\tilde{i} : \tilde{S} \rightarrow \tilde{M}$ extends continuously to a map $\hat{i} : \widehat{\tilde{S}} \rightarrow \widehat{\tilde{M}}$. Hence the limit set of \tilde{S} is locally connected.

Again, combining the Minsky model with Theorem 10.1, we shall get

Theorem [31]: Suppose that $N^h \in H(M, P)$ is a hyperbolic structure on a pared manifold (M, P) with incompressible boundary $\partial_0 M$. Let M_{gf} denote a geometrically finite hyperbolic structure adapted to (M, P) . Then the map $\tilde{i} : \widetilde{M_{gf}} \rightarrow \widetilde{N^h}$ extends continuously to the boundary $\hat{i} : \widehat{\widetilde{M_{gf}}} \rightarrow \widehat{\widetilde{N^h}}$. If Λ denotes the limit set of \widetilde{M} , then Λ is locally connected.

1.2 History and Present State of the Problem

The first major result that started this entire program was Cannon and Thurston's result [10] for hyperbolic 3-manifolds fibering over the circle with fiber a closed surface group.

This was generalised by Minsky who proved the Cannon-Thurston result for bounded geometry Kleinian closed surface groups [23].

An alternate approach (purely in terms of coarse geometry ignoring all local information) was given by the author in [28] generalising the results

of both Cannon-Thurston and Minsky. We proved the Cannon-Thurston result for hyperbolic 3-manifolds of bounded geometry without parabolics and with freely indecomposable fundamental group. A different approach based on Minsky's work was given by Klarreich [19].

Bowditch [3] [4] proved the Cannon-Thurston result for punctured surface Kleinian groups of bounded geometry. In [29] we gave an alternate proof of Bowditch's results and simultaneously generalised the results of Cannon-Thurston, Minsky, Bowditch, and those of [28] to all 3 manifolds of bounded geometry whose cores are incompressible away from cusps. The proof has the advantage that it reduces to a proof for manifolds without parabolics when the 3 manifold in question has freely indecomposable fundamental group and no accidental parabolics.

McMullen [21] proved the Cannon-Thurston result for punctured torus groups, using Minsky's model for these groups [24]. In [30] we identified a large-scale coarse geometric structure involved in the Minsky model for punctured torus groups (and called it **i-bounded geometry**). *i-bounded geometry* can roughly be regarded as that geometry of ends where the boundary torii of Margulis tubes have uniformly bounded diameter. We gave a proof for models of *i-bounded geometry*. In combination with the methods of [29] this was enough to bring under the same umbrella all known results on Cannon-Thurston maps for 3 manifolds whose cores are incompressible away from cusps. In particular, when (M, P) is the pair $S \times I, \delta S \times I$, for S a punctured torus or four-holed sphere, we gave an alternate proof of McMullen's result [21].

In this paper, we define *amalgamation geometry* and prove the Cannon-Thurston result for models of *amalgamation geometry*. We then weaken this assumption to what we call *split geometry* and prove the Cannon-Thurston property for such geometries. In [31] we shall show that the Minsky model for general simply or totally degenerate surface groups [25] [8] gives rise to a model of *split geometry*. This will allow us to conclude that all surface groups have the Cannon-Thurston property and hence have locally connected limit sets. In the sequel to this paper [31], we show that the Minsky model for surface groups has split geometry. This proves that surface groups (and more generally Kleinian groups corresponding to manifolds whose cores are incompressible away from cusps) have locally-connected limit sets.

1.3 Scheme and Outline of the Paper

We first describe in brief, the philosophy of the proof. Given a simply degenerate surface (S) group (without accidental parabolics), Thurston [35] proves that a unique ending lamination λ exists. Let $M = \mathbf{H}^3/\rho(\pi_1(S))$. In this situation, it follows from [35] that any sequence of simple closed curves σ_i , whose geodesic realizations exit the end, converges to λ . Dual to λ , there exists an \mathbb{R} -tree \mathcal{T} and a free action of $\pi_1(S)$ on \mathcal{T} . Now, each σ_i gives rise to a splitting of $\pi_1(S)$, and hence an action of $\pi_1(S)$ on a simplicial tree \mathcal{T}_i . The sequence of these actions converges to the action of $\pi_1(S)$ on \mathcal{T} dual to λ (see for instance, [32]).

The guiding motif of this paper is to find geometric realizations of this sequence of splittings in terms of contiguous blocks B_i (each homeomorphic to $S \times I$). By a geometric realization of a splitting we mean the following: Margulis tubes T_i are chosen, exiting the end of M . Let σ_i denote the core geodesic of T_i . We require that T_i splits some block B_i , i.e. $B_i \setminus T_i$ is homeomorphic to $(S \setminus A(\sigma_i)) \times I$, where $A(\sigma_i)$ is an annular neighborhood of a geodesic representative of σ_i on S . We require further control on the geometry of the complementary pieces $(S \setminus A(\sigma_i)) \times I$.

For conceptual simplicity, assume the T_i 's are separating. Different degrees of control on the geometry of the pieces $(S \setminus A(\sigma_i)) \times I$ give rise to different geometries. Fix a piece of $(S \setminus A(\sigma_i)) \times I$ and call it K . It is better to look at the universal cover \widetilde{B}_i and a lift $\widetilde{K} \subset \widetilde{B}_i$. We adjoin the lifts of T_i bounding \widetilde{K} to \widetilde{K} and call it K_1 . K_1 shall be referred to as a **component** of the relevant geometry.

1) Amalgamation Geometry: The simplest geometry arising from this situation is the case where all K_1 's are uniformly quasiconvex *in the hyperbolic metric on \widetilde{M}* . This is called amalgamation geometry, and can in brief be described as the geometry in which all components are uniformly (hyperbolically) quasiconvex.

2) Graph Amalgamation Geometry: Amalgamation geometry is too restrictive. As a first step towards relaxing this hypothesis, we do not demand that the convex hulls $CH(K_1)$'s be contained in uniformly bounded neighborhoods of the respective K_1 's in the hyperbolic metric. Instead we construct an auxiliary metric called the **graph-metric**. Roughly speaking, the *graph-metric* is the natural simplicial metric on the nerve of the covering of \widetilde{M} by the components K_1 . *Graph Amalgamation Geometry* is the condition that the convex hulls $CH(K_1)$'s lie in uniformly bounded neighborhood of K_1 's in the *graph metric*.

3) Split Geometry: So far, we have assumed that each Margulis tube T_i is contained wholly in a block B_i , splitting it. However, as was pointed out to the author by Yair Minsky and Dick Canary, this is not the most general situation. The T_i 's may *interlock*. To take care of this situation, we allow each tube T_i to cut through (partly or wholly) a uniformly bounded number of blocks. The notions of complementary components and the graph metric still make sense. The resulting geometry is termed *split geometry*.

We shall take one step at a time in this paper, relaxing the hypothesis in the order above. The additional arguments to be introduced as we proceed from one geometry to the next (more general) one will be described as modifications of the core argument relevant to amalgamation geometry.

In the sequel [31], we shall show that simply degenerate ends of hyperbolic 3-manifolds enjoy split geometry.

Outline: A brief outline of the paper follows. Section 2 deals with preliminaries. We also define *amalgamation geometry* via the construction of a model manifold.

Section 3 deals with relative hyperbolicity *a la* Gromov [15], Farb [12] and Bowditch [2].

As in [27], [28], [29], [30], a crucial part of our proof proceeds by constructing a *ladder-like* set $B_\lambda \subset \widetilde{M}$ from a geodesic segment $\lambda \subset \widetilde{S}$ and then a retraction Π_λ of \widetilde{M} onto B_λ .

In Section 4, we construct a model geometry for the universal covers of building blocks and the relevant geometries (electric and graph models) that will concern us.

We also construct the paths that go to build up the *ladder-like set* B_λ . We further construct the restriction of the retraction Π_λ to blocks and show that the retraction does not increase distances much.

In Section 5, we put the blocks and retractions together (by adding them one on top of another) to build the *ladder-like* B_λ and prove the main technical theorem - the existence of a retract Π_λ of \widetilde{M} onto B_λ . This shows that B_λ is quasiconvex in \widetilde{M} equipped with a model pseudometric.

In Section 6, we put together the ingredients from Sections 2, 3, 4 and 5 to prove the existence of a Cannon-Thurston map for simply or doubly degenerate Kleinian groups corresponding to representations of *closed* surface groups that have *amalgamation geometry*.

In Section 7, we extend these results to include surface groups with punctures.

In Section 8, we weaken the hypothesis of *amalgamation geometry* to what we have called *graph amalgamation geometry* and describe the modifications necessary to extend our results to such geometries.

In Section 9, we weaken the hypothesis further to *split geometry* which allows for Margulis tubes to cut across the blocks.

In Section 10, we further generalise these result to include hyperbolic manifolds whose cores are incompressible away from cusps. (We had termed such manifolds *pared manifolds with incompressible boundary* in [29].)

In Section 11, we give a scheme for proving that the Minsky model for surface groups [25] has split geometry. Details will appear in the second part of this paper [31].

In Section 12, we propose an extension of the Sullivan-McMullen dictionary between Kleinian groups and complex dynamics, and suggest an analogue of Yoccoz puzzles in the 3 dimensional setting.

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2 Preliminaries and Amalgamation Geometry

2.1 Hyperbolic Metric Spaces

We start off with some preliminaries about hyperbolic metric spaces in the sense of Gromov [15]. For details, see [11], [14]. Let (X, d) be a hyperbolic metric space. The **Gromov boundary** of X , denoted by ∂X , is the collection of equivalence classes of geodesic rays $r : [0, \infty) \rightarrow X$ with $r(0) = x_0$ for some fixed $x_0 \in X$, where rays r_1 and r_2 are equivalent if $\sup\{d(r_1(t), r_2(t))\} < \infty$. Let $\hat{X} = X \cup \partial X$ denote the natural compactification of X topologized the usual way (cf. [14] pg. 124).

Definitions: A subset Z of X is said to be **k -quasiconvex** if any geodesic joining points of Z lies in a k -neighborhood of Z . A subset Z is **quasiconvex** if it is k -quasiconvex for some k . (For simply connected real hyperbolic manifolds this is equivalent to saying that the convex hull of the set Z lies in a bounded neighborhood of Z . We shall have occasion to use

this alternate characterisation.) A map f from one metric space (Y, d_Y) into another metric space (Z, d_Z) is said to be a (K, ϵ) -**quasi-isometric embedding** if

$$\frac{1}{K}(d_Y(y_1, y_2)) - \epsilon \leq d_Z(f(y_1), f(y_2)) \leq Kd_Y(y_1, y_2) + \epsilon$$

If f is a quasi-isometric embedding, and every point of Z lies at a uniformly bounded distance from some $f(y)$ then f is said to be a **quasi-isometry**. A (K, ϵ) -quasi-isometric embedding that is a quasi-isometry will be called a (K, ϵ) -quasi-isometry.

A (K, ϵ) -**quasigeodesic** is a (K, ϵ) -quasi-isometric embedding of a closed interval in \mathbb{R} . A (K, K) -quasigeodesic will also be called a K -quasigeodesic.

Let (X, d_X) be a hyperbolic metric space and Y be a subspace that is hyperbolic with the inherited path metric d_Y . By adjoining the Gromov boundaries ∂X and ∂Y to X and Y , one obtains their compactifications \widehat{X} and \widehat{Y} respectively.

Let $i : Y \rightarrow X$ denote inclusion.

Definition: Let X and Y be hyperbolic metric spaces and $i : Y \rightarrow X$ be an embedding. A **Cannon-Thurston map** \hat{i} from \widehat{Y} to \widehat{X} is a continuous extension of i .

The following lemma (Lemma 2.1 of [27]) says that a Cannon-Thurston map exists if for all $M > 0$ and $y \in Y$, there exists $N > 0$ such that if λ lies outside an N ball around y in Y then any geodesic in X joining the end-points of λ lies outside the M ball around $i(y)$ in X . For convenience of use later on, we state this somewhat differently.

Lemma 2.1 *A Cannon-Thurston map from \widehat{Y} to \widehat{X} exists if the following condition is satisfied:*

Given $y_0 \in Y$, there exists a non-negative function $M(N)$, such that $M(N) \rightarrow \infty$ as $N \rightarrow \infty$ and for all geodesic segments λ lying outside an N -ball around $y_0 \in Y$ any geodesic segment in Γ_G joining the end-points of $i(\lambda)$ lies outside the $M(N)$ -ball around $i(y_0) \in X$.

The above result can be interpreted as saying that a Cannon-Thurston map exists if the space of geodesic segments in Y embeds properly in the space of geodesic segments in X .

2.2 Amalgamation Geometry

We start with a hyperbolic surface S without punctures. The hyperbolic structure is arbitrary, but it is important that a choice be made.

The Amalgamated Building Block

For the construction of an amalgamated block B , I will denote the closed interval $[0, 3]$. We will describe a geometry on $S \times I$. B has a **geometric core** K with bounded geometry boundary and a preferred geodesic $\gamma (= \gamma_B)$ of bounded length.

There will exist $\epsilon_0, \epsilon_1, D$ (independent of the block B) such that the following hold:

1. B is identified with $S \times I$
2. B has a **geometric core** K identified with $S \times [1, 2]$. (K , in its intrinsic path metric, may be thought of, for convenience, as a convex hyperbolic manifold with boundary consisting of pleated surfaces. But we will have occasion to use geometries that are only quasi-isometric to such geometries when lifted to universal covers. As of now, we do not impose any further restriction on the geometry of K .)
3. γ is homotopic to a simple closed curve on $S \times \{i\}$ for any $i \in I$
4. γ is small, i.e. the length of γ is bounded above by ϵ_0
5. The intrinsic metric on $S \times i$ (for $i = 1, 2$) has bounded geometry, i.e. any closed geodesic on $S \times \{i\}$ has length bounded below by ϵ_1 . Further, the diameter of $S \times \{i\}$ is bounded above by D . (The latter restriction would have followed naturally had we assumed that the curvature of $S \times \{i\}$ is hyperbolic or at least pinched negative.)
6. There exists a regular neighborhood $N_k(\gamma) \subset K$ of γ which is homeomorphic to a solid torus, such that $N_k(\gamma) \cap S \times \{i\}$ is homeomorphic to an open annulus for $i = 1, 2$. We shall have occasion to denote $N_k(\gamma)$ by T_γ and call it the Margulis tube corresponding to γ .
7. $S \times [0, 1]$ and $S \times [1, 2]$ are given the product structures corresponding to the bounded geometry structures on $S \times \{i\}$, for $i = 1, 2$ respectively.

We next describe the geometry of the *geometric core* K . $K - T_\gamma$ has one or two components according as γ does not or does separate S . These components shall be called **amalgamation components** of K . Let K_1 denote such an *amalgamation component*. Then a lift \widetilde{K}_1 of K_1 to \widetilde{K} is bounded by lifts \widetilde{T}_γ of T_γ . The union of such a lift \widetilde{K}_1 along with the lifts \widetilde{T}_γ that bound it will be called an **amalgamation component** of \widetilde{K} .

Note that two amalgamation components of \widetilde{K} , if they intersect, shall do so along a lift \widetilde{T}_γ of T_γ . In this case, they shall be referred to as **adjacent amalgamation components**.

In addition to the above structure of B , we require in addition that there exists $C > 0$ (independent of B) such that

- Each amalgamation component of \widetilde{K} is C -quasiconvex in the intrinsic metric on \widetilde{K} .

Note 1: Quasiconvexity of an amalgamation component follows from the fact that any geometric subgroup of infinite index in a surface group is quasiconvex in the latter. The restriction above is therefore to ensure uniform quasiconvexity. We shall strengthen this restriction further when we describe the geometry of \widetilde{M} , where M is a 3-manifold built up of blocks of *amalgamation geometry* and those of bounded geometry by gluing them end to end. We shall require that each amalgamation component is **uniformly quasiconvex in \widetilde{M}** rather than just in \widetilde{K} .

Note 2: So far, the restrictions on K are quite mild. There are really two restrictions. One is the existence of a bounded length simple closed geodesic whose regular neighborhood intersects the bounding surfaces of K in annuli. The second restriction is that the two boundary surfaces of K have *bounded geometry*.

The copy of $S \times I$ thus obtained, with the restrictions above, will be called a **building block of amalgamated geometry** or an **amalgamation geometry building block**, or simply an **amalgamation block**.

Thick Block

Fix constants D, ϵ and let $\mu = [p, q]$ be an ϵ -thick Teichmüller geodesic of length less than D . μ is ϵ -thick means that for any $x \in \mu$ and any closed geodesic η in the hyperbolic surface S_x over x , the length of η is greater than ϵ . Now let B denote the universal curve over μ reparametrized such that the length of μ is covered in unit time. Thus $B = S \times [0, 1]$ topologically.

B is given the path metric and is called a **thick building block**.

Note that after acting by an element of the mapping class group, we might as well assume that μ lies in some given compact region of Teichmüller

space. This is because the marking on $S \times \{0\}$ is not important, but rather its position relative to $S \times \{1\}$. Further, since we shall be constructing models only upto quasi-isometry, we might as well assume that $S \times \{0\}$ and $S \times \{1\}$ *lie in the orbit* under the mapping class group of some fixed base surface. Hence μ can be further simplified to be a Teichmuller geodesic joining a pair (p, q) amongst a finite set of points in the orbit of a fixed hyperbolic surface S .

The Model Manifold

Note that the boundary of an amalgamation block B_i consists of $S \times \{0, 3\}$ and the intrinsic path metric on each such $S \times \{0\}$ or $S \times \{3\}$ is of bounded geometry. Also, the boundary of a thick block B consists of $S \times \{0, 1\}$, where S_0, S_1 lie in some given bounded region of Teichmuller space. The intrinsic path metrics on each such $S \times \{0\}$ or $S \times \{1\}$ is the path metric on S .

The model manifold of **amalgamation geometry** is obtained from $S \times J$ (where J is a sub-interval of \mathbb{R} , which may be semi-infinite or bi-infinite. In the former case, we choose the usual normalisation $J = [0, \infty)$) by first choosing a sequence of blocks B_i (thick or amalgamated) and corresponding intervals $I_i = [0, 1]$ or $[0, 3]$ according as B_i is thick or amalgamated. The metric on $S \times I_i$ is then declared to be that on the building block B_i . Implicitly, we are requiring that the surfaces along which gluing occurs have the same metric. Thus we have,

Definition: A manifold M homeomorphic to $S \times J$, where $J = [0, \infty)$ or $J = (-\infty, \infty)$, is said to be a model of **amalgamation geometry** if

1. there is a fiber preserving homeomorphism from M to $\tilde{S} \times J$ that lifts to a quasi-isometry of universal covers
2. there exists a sequence I_i of intervals (with disjoint interiors) and blocks B_i where the metric on $S \times I_i$ is the same as that on some building block B_i
3. $\bigcup_i I_i = J$
4. There exists $C > 0$ such that for all amalgamated blocks B and geometric cores $K \subset B$, all amalgamation components of \tilde{K} are C -quasiconvex in \tilde{M}

Note: The last restriction (4) above is a global restriction on the geometry of amalgamation components, not just a local one (i.e. quasiconvexity in \widetilde{M} rather than \widetilde{B} is required.)

The figure below illustrates schematically what the model looks like. Filled squares correspond to solid torii along which amalgamation occurs. The adjoining piece(s) denote amalgamation blocks of K . The blocks which have no filled squares are the *thick blocks* and those with filled squares are the *amalgamated blocks*

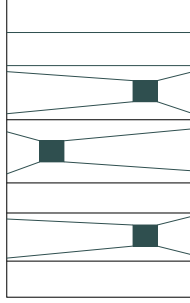


Figure 1: *Model of amalgamated geometry (schematic)*

Definition: A manifold M homeomorphic to $S \times J$, where $J = [0, \infty)$ or $J = (-\infty, \infty)$, is said to have **amalgamated geometry** if there exists $K, \epsilon > 0$ and a model manifold M_1 of *amalgamation geometry* such that

1. there exists a homeomorphism ϕ from M to M_1 . This induces from the block decomposition of M_1 a block decomposition of M .
2. We require in addition that the induced homeomorphism $\tilde{\phi}$ between universal covers of blocks is a (K, ϵ) quasi-isometry.

We shall usually suppress the homeomorphism ϕ and take M itself to be a model manifold of *amalgamation geometry*.

A geometrically tame hyperbolic 3-manifold is said to have **amalgamated geometry** if each end has amalgamated geometry.

Note: We shall later have occasion to introduce a different model, called the **graph model**

3 Relative Hyperbolicity

In this section, we shall recall first certain notions of relative hyperbolicity due to Farb [12], Klarreich [19] and the author [30]. Using these, we shall derive certain Lemmas that will be useful in studying the geometry of the universal covers of building blocks.

3.1 Electric Geometry

We start with a surface S (assumed hyperbolic for the time being) of (K, ϵ) bounded geometry, i.e. S has diameter bounded by K and injectivity radius bounded below by ϵ . Let σ be a simple closed geodesic on S . Replace σ by a copy of $\sigma \times [0, 1]$, by cutting open along σ and gluing in a copy of $\sigma \times [0, 1] = A_\sigma$. (This is like ‘grafting’ but we shall not have much use for this similarity in this paper.) Let S_G denote the grafted surface. $S_G - A_\sigma$ has one or two components according as σ does not or does separate S . Call these **amalgamation component(s)** of S . We shall denote amalgamation components as S_A . We construct a pseudometric on S_G , by declaring the metric on each amalgamation component to be zero and to be the product metric on A_σ . Thus we define:

- the length of any path that lies in the interior of an amalgamation component to be zero
- the length of any path that lies in A_σ to be its (Euclidean) length in the path metric on A_σ
- the length of any other path to be the sum of lengths of pieces of the above two kinds.

This allows us to define distances by taking the infimum of lengths of paths joining pairs of points and gives us a path pseudometric, which we call the **electric metric** on S_G . The electric metric also allows us to define geodesics. Let us call S_G equipped with the above pseudometric (S_{Gel}, d_{Gel}) (to be distinguished from a ‘dual’ construction of an electric metric S_{el} used in [30], where the geodesic σ , rather than its complementary component(s) is electrocuted.)

Important Note: We may and shall regard S as a graph of groups with vertex group(s) the subgroup(s) corresponding to amalgamation component(s) and edge group Z , the fundamental group of A_σ . Then \tilde{S} equipped with the lift of the above pseudometric is quasi-isometric to the tree corresponding to the splitting on which $\pi_1(S)$ acts.

We shall be interested in the universal cover $\widetilde{S_{Gel}}$ of S_{Gel} . Paths in S_{Gel} and $\widetilde{S_{Gel}}$ will be called electric paths (following Farb [12]). Geodesics and quasigeodesics in the electric metric will be called electric geodesics and electric quasigeodesics respectively.

Definitions:

- A path $\gamma : I \rightarrow Y$ in a path metric space Y is a K -quasigeodesic if we have

$$L(\beta) \leq KL(A) + K$$

for any subsegment $\beta = \gamma|_{[a,b]}$ and any rectifiable path $A : [a,b] \rightarrow Y$ with the same endpoints.

- γ is said to be an electric K, ϵ -quasigeodesic in $\widetilde{S_{Gel}}$ **without backtracking** if γ is an electric K -quasigeodesic in $\widetilde{S_{Gel}}$ and γ does not return to any lift $\widetilde{S_A} \subset \widetilde{S_{Gel}}$ (of an amalgamation component $S_A \subset S$) after leaving it.

We collect together certain facts about the electric metric that Farb proves in [12]. $N_R(Z)$ will denote the R -neighborhood about the subset Z in the hyperbolic metric. $N_R^e(Z)$ will denote the R -neighborhood about the subset Z in the electric metric.

Lemma 3.1 (*Lemma 4.5 and Proposition 4.6 of [12]*)

1. *Electric quasi-geodesics electrically track hyperbolic geodesics: Given $P > 0$, there exists $K > 0$ with the following property: For some $\widetilde{S_{Gel}}$, let β be any electric P -quasigeodesic without backtracking from x to y , and let γ be the hyperbolic geodesic from x to y . Then $\beta \subset N_K^e(\gamma)$.*
2. *Hyperbolicity: There exists δ such that each $\widetilde{S_{Gel}}$ is δ -hyperbolic, independent of the curve σ whose lifts are electrocuted.*

Note: As pointed out before, S_{Gel} is quasi-isometric to a tree and is therefore hyperbolic. The above assertion holds in far greater generality than stated. We discuss this below.

We consider a hyperbolic metric space X and a collection \mathcal{H} of (*uniformly*) C -*quasiconvex uniformly separated subsets*, i.e. there exists $D > 0$ such that for $H_1, H_2 \in \mathcal{H}$, $d_X(H_1, H_2) \geq D$. In this situation X is hyperbolic relative to the collection \mathcal{H} . The result in this form is due to Klarreich [19]. We give the general version of Farb's theorem below and refer to [12] and Klarreich [19] for proofs.

Lemma 3.2 (See Lemma 4.5 and Proposition 4.6 of [12] and Theorem 5.3 of Klarreich [19]) Given δ, C, D there exists Δ such that if X is a δ -hyperbolic metric space with a collection \mathcal{H} of C -quasiconvex D -separated sets. then,

1. *Electric quasi-geodesics electrically track hyperbolic geodesics:* Given $P > 0$, there exists $K > 0$ with the following property: Let β be any electric P -quasigeodesic from x to y , and let γ be the hyperbolic geodesic from x to y . Then $\beta \subset N_K^e(\gamma)$.
2. γ lies in a hyperbolic K -neighborhood of $N_0(\beta)$, where $N_0(\beta)$ denotes the zero neighborhood of β in the electric metric.
3. *Hyperbolicity:* X is Δ -hyperbolic.

A special kind of *geodesic without backtracking* will be necessary for universal covers $\widetilde{S_{Gel}}$ of surfaces with some electric metric. Let σ, A_σ be as before.

Let λ_e be an electric geodesic in some $(\widetilde{S_{Gel}}, d_{Gel})$. Then, each segment of λ_e between two lifts \widetilde{A}_σ of A_σ (i.e. lying inside a lift of an amalgamation component) is required to be perpendicular to the bounding geodesics. We shall refer to these segments of λ_e as **amalgamation segments** because they lie inside lifts of the amalgamation components.

Let a, b be the points at which λ_e enters and leaves a lift \widetilde{A}_σ of A_σ . If a, b lie on the same side, i.e. on a lift of either $\sigma \times \{0\}$ or $\sigma \times \{1\}$, then we join a, b by the geodesic joining them. If they lie on opposite sides of \widetilde{A}_σ , then assume, for convenience, that a lies on a lift of $\sigma \times \{0\}$ and b lies on a lift of $\sigma \times \{1\}$. Then we join a to b by a union of 2 geodesic segments $[a, c]$ and $[d, b]$ lying along $\widetilde{\sigma} \times \{0\}$ and $\widetilde{\sigma} \times \{1\}$ respectively (for some lift \widetilde{A}_σ), along with a ‘horizontal’ segment $[c, d]$, where $[c, d] \subset \widetilde{A}_\sigma$ projects to a segment of the form $\{x\} \times [0, 1] \subset \sigma \times [0, 1]$. We further require that the sum of the lengths $d(a, c)$ and $d(d, b)$ is the minimum possible. The union of the three segments $[a, c], [c, d], [d, b]$ shall be denoted by $[a, b]_{int}$ and shall be referred to as an **interpolating segment**. See figure below.

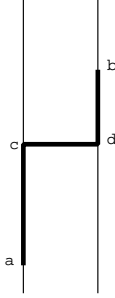


Figure 2: *Interpolating segment*

The union of the *amalgamation segments* along with the *interpolating segments* gives rise to a preferred representative of a quasigeodesic without backtracking joining the end-points of λ_{Gel} . Such a representative of the class of λ_{Gel} shall be called the **canonical representative** of λ_{Gel} . Further, the underlying set of the canonical representative in the *hyperbolic metric* shall be called the **electro-ambient representative** λ_q of λ_e . Since λ_q turns out to be a hyperbolic quasigeodesic (Lemma 3.4 below), we shall also call it an **electro-ambient quasigeodesic**. See Figure 3 below:

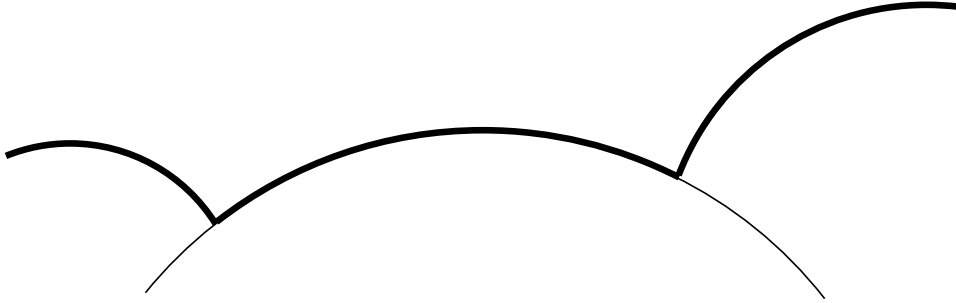


Figure 3: *Electro-ambient quasigeodesic*

Remark: We note first that if we collapse each lift of A_σ along the $I(=[0, 1])$ -fibres, (and thus obtain a geodesic that is a lift of σ), then λ_{Gel} becomes an electric geodesic λ_{el} in the universal cover \widetilde{S}_{el} of S_{el} . Here S_{el} denotes the space obtained by electrocuting the geodesic σ (See Section 3.1 of [30]).

Let $c : S_G \rightarrow S$ be the map that *collapses* I -fibres, i.e. it maps the annulus $A_\sigma = \sigma \times I$ to the geodesic σ by taking (x, t) to x . The lift $\tilde{c} : \tilde{S}_G \rightarrow \tilde{S}$

collapses each lift of A_σ along the $I(=[0, 1])$ -fibres to a geodesic that is a lift of σ). Also it takes λ_{Gel} to an electric geodesic λ_{el} in the universal cover $\widetilde{S_{el}}$ of S_{el} (that λ_{el} is an electric geodesic in $\widetilde{S_{el}}$ follows easily, say from normal forms). These were precisely the electro-ambient quasigeodesics in the space $\widetilde{S_{el}}$ (See Section 3.1 of [30] for definitions).

Remark: The electro-ambient geodesics in the sense of [30] and those in the present paper differ slightly. The difference is due to the grafting annulus A_σ that we use here in place of σ . What is interesting is that whether we electrocute σ (to obtain S_{el}) or its complementary components (to obtain S_{Gel}), we obtain very nearly the same electro-ambient geodesics. In fact modulo c , they *are* the same.

We now recall a Lemma from [30]:

Lemma 3.3 (See Lemma 3.7 of [30]) *There exists (K, ϵ) such that each electro-ambient representative λ_{el} of an electric geodesic in $\widetilde{S_{el}}$ is a (K, ϵ) hyperbolic quasigeodesic.*

Since \tilde{c} is clearly a quasi-isometry, it follows easily that:

Lemma 3.4 *There exists (K, ϵ) such that each electro-ambient representative λ_{Gel} of an electric geodesic in $\widetilde{S_{Gel}}$ is a (K, ϵ) hyperbolic quasigeodesic.*

In the above form, *electro-ambient quasigeodesics* are considered only in the context of surfaces, closed geodesics on them and their complementary (*amalgamation*) components. A considerable generalisation of this was obtained in [30], which will be necessary while considering the global geometry of \widetilde{M} (rather than the geometry of \widetilde{B} , for an amalgamated building block B).

We recall a definition from [30]:

Definitions: Given a collection \mathcal{H} of C -quasiconvex, D -separated sets and a number ϵ we shall say that a geodesic (resp. quasigeodesic) γ is a geodesic (resp. quasigeodesic) **without backtracking** with respect to ϵ neighborhoods if γ does not return to $N_\epsilon(H)$ after leaving it, for any $H \in \mathcal{H}$. A geodesic (resp. quasigeodesic) γ is a geodesic (resp. quasigeodesic) **without backtracking** if it is a geodesic (resp. quasigeodesic) without backtracking with respect to ϵ neighborhoods for some $\epsilon \geq 0$.

Note: For strictly convex sets, $\epsilon = 0$ suffices, whereas for convex sets any $\epsilon > 0$ is enough.

Let X be a δ -hyperbolic metric space, and \mathcal{H} a family of C -quasiconvex, D -separated, collection of subsets. Then by Lemma 3.2, X_{el} obtained by electrocuting the subsets in \mathcal{H} is a $\Delta = \Delta(\delta, C, D)$ -hyperbolic metric space. Now, let $\alpha = [a, b]$ be a hyperbolic geodesic in X and β be an electric P -quasigeodesic without backtracking joining a, b . Replace each maximal subsegment, (with end-points p, q , say) starting from the left of β lying within some $H \in \mathcal{H}$ by a hyperbolic geodesic $[p, q]$. The resulting **connected** path β_q is called an *electro-ambient representative* in X .

In [30] we noted that β_q need not be a hyperbolic quasigeodesic. However, we did adapt Proposition 4.3 of Klarreich [19] to obtain the following:

Lemma 3.5 (See Proposition 4.3 of [19], also see Lemma 3.10 of [30])
*Given δ, C, D, P there exists C_3 such that the following holds:
 Let (X, d) be a δ -hyperbolic metric space and \mathcal{H} a family of C -quasiconvex, D -separated collection of quasiconvex subsets. Let (X, d_e) denote the electric space obtained by electrocuting elements of \mathcal{H} . Then, if α, β_q denote respectively a hyperbolic geodesic and an electro-ambient P -quasigeodesic with the same end-points, then α lies in a (hyperbolic) C_3 neighborhood of β_q .*

Note: The above Lemma will be needed while considering geodesics in \widetilde{M} .

3.2 Electric isometries

Recall that S_G is a grafted surface obtained from a (fixed) hyperbolic metric by grafting an annulus A_σ in place of a geodesic σ .

Now let ϕ be any diffeomorphism of S_G that fixes A_σ pointwise and (in case $(S_G - A_\sigma)$ has two components) preserves each amalgamation component as a set, i.e. ϕ sends each amalgamation component to itself. Such a ϕ will be called a **component preserving diffeomorphism**. Then in the electrocuted surface S_{Gel} , any electric geodesic has length equal to the number of times it crosses A_σ . It follows that ϕ is an isometry of S_{Gel} . (See Lemma 3.12 of [30] for an analogous result in S_{el} .) We state this below.

Lemma 3.6 *Let ϕ denote a component preserving diffeomorphism of S_G . Then ϕ induces an isometry of (S_{Gel}, d_{Gel}) .*

Everything in the above can be lifted to the universal cover $\widetilde{S_{Gel}}$. We let $\tilde{\phi}$ denote the lift of ϕ to $\widetilde{S_{Gel}}$. This gives

Lemma 3.7 *Let $\tilde{\phi}$ denote a lift of a component preserving diffeomorphism ϕ to $(\widetilde{S_{Gel}}, d_{Gel})$. Then $\tilde{\phi}$ induces an isometry of $(\widetilde{S_{Gel}}, d_{Gel})$.*

3.3 Nearest-point Projections

We need the following basic lemmas from [28] and [30].

The following Lemma says nearest point projections in a δ -hyperbolic metric space do not increase distances much.

Lemma 3.8 *(Lemma 3.1 of [28]) Let (Y, d) be a δ -hyperbolic metric space and let $\mu \subset Y$ be a C -quasiconvex subset, e.g. a geodesic segment. Let $\pi : Y \rightarrow \mu$ map $y \in Y$ to a point on μ nearest to y . Then $d(\pi(x), \pi(y)) \leq C_3 d(x, y)$ for all $x, y \in Y$ where C_3 depends only on δ, C .*

The next lemma says that quasi-isometries and nearest-point projections on hyperbolic metric spaces ‘almost commute’.

Lemma 3.9 *(Lemma 3.5 of [28]) Suppose (Y_1, d_1) and (Y_2, d_2) are δ -hyperbolic. Let μ_1 be some geodesic segment in Y_1 joining a, b and let p be any vertex of Y_1 . Also let q be a vertex on μ_1 such that $d_1(p, q) \leq d_2(p, x)$ for $x \in \mu_1$. Let ϕ be a (K, ϵ) - quasiisometric embedding from Y_1 to Y_2 . Let μ_2 be a geodesic segment in Y_2 joining $\phi(a)$ to $\phi(b)$. Let r be a point on μ_2 such that $d_2(\phi(p), r) \leq d_2(\phi(p), x)$ for $x \in \mu_2$. Then $d_2(r, \phi(q)) \leq C_4$ for some constant C_4 depending only on K, ϵ and δ .*

For our purposes we shall need the above Lemma for quasi-isometries from $\widetilde{S_a}$ to $\widetilde{S_b}$ for two different hyperbolic structures on the same surface. We shall also need it for electrocuted surfaces.

Yet another property that we shall require for nearest point projections is that nearest point projections in the electric metric and in the ‘almost hyperbolic’ metric (coming as a lift of the metric on S_G) almost agree. Let $\widetilde{S_G} = Y$ be the universal cover of a surface with the grafted metric. Equip Y with the path metric d as usual. Then Y is quasi-isometric to the hyperbolic plane. Recall that d_{Gel} denotes the electric metric on Y obtained by electrocuting the lifts of complementary components. Now, let $\mu = [a, b]$ be a geodesic on (Y, d) and let μ_q denote the electro-ambient quasigeodesic joining a, b (See Lemma 3.3). Let π denote the nearest point projection in (Y, d) . Tentatively, let π_e denote the nearest point projection in (Y, d_{Gel}) . Note that π_e is not well-defined. It is defined upto a bounded amount of

discrepancy in the electric metric d_e . But we would like to make π_e well-defined upto a bounded amount of discrepancy in the metric d .

Definition: Let $y \in Y$ and let μ_q be an electro-ambient representative of an electric geodesic μ_{Gel} in (Y, d_{Gel}) . Then $\pi_e(y) = z \in \mu_q$ if the ordered pair $\{d_{Gel}(y, \pi_e(y)), d(y, \pi_e(y))\}$ is minimised at z .

The proof of the following Lemma shows that this gives us a definition of π_e which is ambiguous by a finite amount of discrepancy not only in the electric metric but also in the hyperbolic metric.

Lemma 3.10 *There exists $C > 0$ such that the following holds. Let μ be a hyperbolic geodesic joining a, b . Let μ_{Gel} be an electric geodesic joining a, b . Also let μ_q be the electro-ambient representative of μ_{Gel} . Let π_h denote the nearest point projection of Y onto μ . $d(\pi_h(y), \pi_e(y))$ is uniformly bounded.*

Proof: This Lemma is similar to Lemma 3.16 of [30], but its proof is somewhat different. For the purposes of this lemma we shall refer to the metric on \widetilde{S}_G as the hyperbolic metric whereas it is in fact only quasi-isometric to it.

$[u, v]$ and $[u, v]_q$ will denote respectively the hyperbolic geodesic and the electro-ambient quasigeodesic joining u, v . Since $[u, v]_q$ is a quasigeodesic by Lemma 3.3, it suffices to show that for any y , its hyperbolic and electric projections $\pi_h(y), \pi_e(y)$ almost agree.

First note that any hyperbolic geodesic η in \widetilde{S}_G is also an electric geodesic. This follows from the fact that $(\widetilde{S}_G, d_{Gel})$ maps to a tree T (arising from the splitting along σ) with the pullback of every vertex a set of diameter zero in the pseudometric d_{Gel} . Now if a path in \widetilde{S}_G projects to a path in T that is not a geodesic, then it must backtrack. Hence, it must leave an amalgamating component and return to it. Such a path can clearly not be a hyperbolic geodesic in \widetilde{S}_G (since each amalgamating component is convex).

Next, it follows that hyperbolic projections automatically minimise electric distances. Else as in the preceding paragraph, $[y, \pi_h(y)]$ would have to cut a lift of $\widetilde{\sigma} = \widetilde{\sigma}_1$ that separates $[u, v]_q$. Further, $[y, \pi_h(y)]$ cannot return to $\widetilde{\sigma}_1$ after leaving it. Let z be the first point at which $[y, \pi_h(y)]$ meets $\widetilde{\sigma}_1$. Also let w be the point on $[u, v]_q \cap \widetilde{\sigma}_1$ that is nearest to z . Since amalgamation segments of $[u, v]_q$ meeting $\widetilde{\sigma}_1$ are perpendicular to the latter, it follows that $d(w, z) < d(w, \pi_h(y))$ and therefore $d(y, z) < d(y, \pi_h(y))$ contradicting the definition of $\pi_h(y)$. Hence hyperbolic projections automatically minimise electric distances.

Further, it follows by repeating the argument in the first paragraph that $[y, \pi_h(y)]$ and $[y, \pi_e(y)]$ pass through the same set of amalgamation components in the same order; in particular they cut across the same set of lifts of $\tilde{\sigma}$. Let $\tilde{\sigma}_2$ be the last such lift. Then $\tilde{\sigma}_2$ forms the boundary of an amalgamation component \tilde{S}_A whose intersection with $[u, v]_q$ is of the form $[a, b] \cup [b, c] \cup [c, d]$, where $[a, b] \subset \tilde{\sigma}_3$ and $[c, d] \subset \tilde{\sigma}_4$ are subsegments of two lifts of σ and $[b, c]$ is perpendicular to these two. Then the nearest-point projection of $\tilde{\sigma}_2$ onto each of $[a, b]$, $[b, c]$, $[c, d]$ has uniformly bounded diameter. Hence the nearest point projection of $\tilde{\sigma}_2$ onto the hyperbolic geodesic $[a, d] \subset \tilde{S}_A$ has uniformly bounded diameter. The result follows. \square

3.4 Coboundedness and Consequences

In this Section, we collect together a few more results that strengthen Lemmas 3.1 and 3.2.

Definition: A collection \mathcal{H} of uniformly C -quasiconvex sets in a δ -hyperbolic metric space X is said to be **mutually D-cobounded** if for all $H_i, H_j \in \mathcal{H}$, $\pi_i(H_j)$ has diameter less than D , where π_i denotes a nearest point projection of X onto H_i . A collection is **mutually cobounded** if it is mutually D-cobounded for some D .

Lemma 3.11 *Suppose X is a δ -hyperbolic metric space with a collection \mathcal{H} of C -quasiconvex K -separated D -mutually cobounded subsets. There exists $\epsilon_0 = \epsilon_0(C, K, D, \delta)$ such that the following holds:*

Let β be an electric P -quasigeodesic without backtracking and γ a hyperbolic geodesic, both joining x, y . Then, given $\epsilon \geq \epsilon_0$ there exists $D = D(P, \epsilon)$ such that

1. *Similar Intersection Patterns 1: if precisely one of $\{\beta, \gamma\}$ meets an ϵ -neighborhood $N_\epsilon(H_1)$ of an electrocuted quasiconvex set $H_1 \in \mathcal{H}$, then the length (measured in the intrinsic path-metric on $N_\epsilon(H_1)$) from the entry point to the exit point is at most D .*
2. *Similar Intersection Patterns 2: if both $\{\beta, \gamma\}$ meet some $N_\epsilon(H_1)$ then the length (measured in the intrinsic path-metric on $N_\epsilon(H_1)$) from the entry point of β to that of γ is at most D ; similarly for exit points.*

Summarizing, we have:

- If X is a hyperbolic metric space and \mathcal{H} a collection of uniformly quasiconvex mutually cobounded separated subsets, then X is hyperbolic relative to the collection \mathcal{H} and satisfies *Bounded Penetration*, i.e. hyperbolic geodesics and electric quasigeodesics have similar intersection patterns in the sense of Lemma 3.11.

The relevance of co-boundedness comes from the following Lemma which is essentially due to Farb [12].

Lemma 3.12 *Let M^h be a hyperbolic manifold of i -bounded geometry, with Margulis tubes $T_i \in \mathcal{T}$ and horoballs $H_j \in \mathcal{H}$. Then the lifts \tilde{T}_i and \tilde{H}_j are mutually co-bounded.*

The proof given in [12] is for a collection of separated horospheres, but the same proof works for neighborhoods of geodesics and horospheres as well.

A closely related theorem was proved by McMullen (Theorem 8.1 of [21]).

As usual, $N_R(Z)$ will denote the R -neighborhood of the set Z .

Let \mathcal{H} be a locally finite collection of horoballs in a convex subset X of \mathbb{H}^n (where the intersection of a horoball, which meets ∂X in a point, with X is called a horoball in X).

Definition: The ϵ -neighborhood of a bi-infinite geodesic in \mathbb{H}^n will be called a **thickened geodesic**.

Theorem 3.13 [21] *Let $\gamma : I \rightarrow X \setminus \bigcup \mathcal{H}$ be an ambient K -quasigeodesic (for X a convex subset of \mathbb{H}^n) and let \mathcal{H} denote a uniformly separated collection of horoballs and thickened geodesics. Let η be the hyperbolic geodesic with the same endpoints as γ . Let $\mathcal{H}(\eta)$ be the union of all the horoballs and thickened geodesics in \mathcal{H} meeting η . Then $\eta \cup \mathcal{H}(\eta)$ is (uniformly) quasiconvex and $\gamma(I) \subset B_R(\eta \cup \mathcal{H}(\eta))$, where R depends only on K .*

4 Universal Covers of Building Blocks and Electric Geometry

4.1 Graph Model of Building Blocks

Amalgamation Blocks

Given a geodesic segment $\lambda \subset \tilde{S}$ and a basic *amalgamation building block* B , let $\lambda = [a, b] \subset \tilde{S} \times \{0\}$ be a geodesic segment, where $\tilde{S} \times \{0\} \subset \tilde{B}$.

We shall now build a graph model for \tilde{B} which will be quasi-isometric to an electrocuted version of the original model, where amalgamation components of the geometric core K are electrocuted.

$\tilde{S} \times \{0\}$ and $\tilde{S} \times \{3\}$ are equipped with hyperbolic metrics. $\tilde{S} \times \{1\}$ and $\tilde{S} \times \{2\}$ are grafted surfaces with electric metric obtained by electrocuting the amalgamation components. This constructs 4 ‘sheets’ of \tilde{S} comprising the ‘horizontal skeleton’ of the ‘graph model’ of \tilde{B} . Now for the vertical strands. On each vertical element of the form $x \times [0, 1]$ and $x \times [2, 3]$ put the Euclidean metric.

To do this precisely, one needs to take a bit more care and perform the construction in the universal cover. For each amalgamation component of \tilde{K} (recall that such a component is a lift of an amalgamation component of K to the universal cover along with bounding lifts \tilde{T}_σ of the Margulis tubes). For each such component \tilde{K}_i we construct $\tilde{K}_i \times [0, 1/2]$, so that any two copies $\tilde{K}_i \times [0, 1/2]$ and $\tilde{K}_j \times [0, 1/2]$ intersect (if at all they do) only along the original bounding lifts \tilde{T}_σ of the Margulis tubes. In particular the copies $\tilde{K}_i \times [0, 1/2]$ intersect \tilde{K} along $\tilde{K}_i \times \{0\}$. Next put the zero metric on each copy of $\tilde{K}_i \times \{1/2\}$.

This construction is very closely related to the ‘coning’ construction introduced by Farb in [12].

The resulting copy of \tilde{B} will be called the **graph model of an amalgamation block**.

Next, we give an I -bundle structure to K that preserves the grafting annulus. Thus $A_\sigma \times [1, 2]$ has a structure of a Margulis tube. Let ϕ denote a map from $S \times \{1\}$ to $S \times \{2\}$ mapping $(x, 1)$ to $(x, 2)$. Clearly there is a bound l_B on the length in K of $x \times [1, 2]$ as x ranges over $S \times \{1\}$. That is to say that the core K has a bounded thickness. This bound depends on the block B we are considering.

Let $\tilde{\phi}$ denote the lift of ϕ to \tilde{K} . Then $\tilde{\phi}$ is a (k, ϵ) -quasi-isometry where k, ϵ depend on the block B .

Thick Block

For a thick block $B = \tilde{S} \times [0, 1]$, recall that B is the universal curve over a ‘thick’ Teichmüller geodesic $\lambda_{Teich} = [a, b]$ of length less than some fixed $D > 0$. Each $S \times \{x\}$ is identified with the hyperbolic surface over $(a + \frac{x}{b-a})$ (assuming that the Teichmüller geodesic is parametrized by arc-length).

Here $S \times \{0\}$ is identified with the hyperbolic surface corresponding to

a , $S \times \{1\}$ is identified with the hyperbolic surface corresponding to b and each (x, a) is joined to (x, b) by a segment of length 1.

The resulting model of \tilde{B} is called a **graph model of a thick block**.

Metrics on graph models are called **graph metrics**.

Admissible Paths

Admissible paths consist of the following :

1. Horizontal segments along some $\tilde{S} \times \{i\}$ for $i = \{0, 1, 2, 3\}$ (amalgamated blocks) or $i = \{0, 1\}$ (thick blocks).
2. Vertical segments $x \times [0, 1]$ or $x \times [2, 3]$ for amalgamated blocks or $x \times [0, 1]$ for thick blocks.
3. Vertical segments of length $\leq l_B$ joining $x \times \{1\}$ to $x \times \{2\}$ for amalgamated blocks.

4.2 Construction of Quasiconvex Sets for Building Blocks

In the next section, we will construct a set B_λ containing λ and a retraction Π_λ of \tilde{M} onto it. Π_λ will have the property that it does not stretch distances much. This will show that B_λ is quasi-isometrically embedded in \tilde{M} .

In this subsection, we describe the construction of B_λ restricted to a building block B .

Construction of $B_\lambda(B)$ - Thick Block

Let the thick block be the universal curve over a Teichmuller geodesic $[\alpha, \beta]$. Let S_α denote the hyperbolic surface over α and S_β denote the hyperbolic surface over β .

First, let $\lambda = [a, b]$ be a geodesic segment in \tilde{S} . Let λ_{B0} denote $\lambda \times \{0\}$.

Next, let ψ be the lift of the 'identity' map from \tilde{S}_α to \tilde{S}_β . Let Ψ denote the induced map on geodesics and let $\Psi(\lambda)$ denote the hyperbolic geodesic joining $\psi(a), \psi(b)$. Let λ_{B1} denote $\Psi(\lambda) \times \{1\}$.

For the universal cover \tilde{B} of the thick block B , define:

$$B_\lambda(B) = \bigcup_{i=0,1} \lambda_{Bi}$$

Definition: Each $\tilde{S} \times i$ for $i = 0, 1$ will be called a **horizontal sheet** of \tilde{B} when B is a thick block.

Construction of $B_\lambda(B)$ - Amalgamation Block

First, recall that $\lambda = [a, b]$ is a geodesic segment in \tilde{S} . Let λ_{B0} denote $\lambda \times \{0\}$.

Next, let λ_{Gel} denote the electric geodesic joining a, b in the electric pseudo-metric on \tilde{S} obtained by electrocuting lifts of σ . Let λ_{B1} denote $\lambda_{Gel} \times \{1\}$.

Third, recall that $\tilde{\phi}$ is the lift of a component preserving diffeomorphism ϕ to \tilde{S} equipped with the electric metric d_{Gel} . Let $\tilde{\Phi}$ denote the induced map on geodesics, i.e. if $\mu = [x, y] \subset (\tilde{S}, d_{Gel})$, then $\tilde{\Phi}(\mu) = [\phi(x), \phi(y)]$ is the geodesic joining $\phi(x), \phi(y)$. Let λ_{B2} denote $\Phi(\lambda_{Gel}) \times \{2\}$.

Fourthly, let $\Phi(\lambda)$ denote the hyperbolic geodesic joining $\phi(a), \phi(b)$. Let λ_{B3} denote $\Phi(\lambda) \times \{3\}$.

For the universal cover \tilde{B} of the thin block B , define:

$$B_\lambda(B) = \bigcup_{i=0, \dots, 3} \lambda_{Bi}$$

Definition: Each $\tilde{S} \times i$ for $i = 0 \dots 3$ will be called a **horizontal sheet** of \tilde{B} when B is a thick block.

Construction of $\Pi_{\lambda, B}$ - Thick Block

On $\tilde{S} \times \{0\}$, let Π_{B0} denote nearest point projection onto λ_{B0} in the path metric on $\tilde{S} \times \{0\}$.

On $\tilde{S} \times \{1\}$, let Π_{B1} denote nearest point projection onto λ_{B1} in the path metric on $\tilde{S} \times \{1\}$.

For the universal cover \tilde{B} of the thick block B , define:

$$\Pi_{\lambda, B}(x) = \Pi_{Bi}(x), x \in \tilde{S} \times \{i\}, i = 0, 1$$

Construction of $\Pi_{\lambda, B}$ - Amalgamation Block

On $\tilde{S} \times \{0\}$, let Π_{B0} denote nearest point projection onto λ_{B0} . Here the nearest point projection is taken in the path metric on $\tilde{S} \times \{0\}$ which is a hyperbolic metric space.

On $\tilde{S} \times \{1\}$, let Π_{B1} denote the nearest point projection onto λ_{B1} . Here the nearest point projection is taken in the sense of the definition preceding Lemma 3.10, i.e. minimising the ordered pair (d_{Gel}, d_{hyp}) (where d_{Gel}, d_{hyp} refer to electric and hyperbolic metrics respectively.)

On $\tilde{S} \times \{2\}$, let Π_{B2} denote the nearest point projection onto λ_{B2} . Here, again the nearest point projection is taken in the sense of the definition preceding Lemma 3.10.

Again, on $\tilde{S} \times \{3\}$, let Π_{B3} denote nearest point projection onto λ_{B3} . Here the nearest point projection is taken in the path metric on $\tilde{S} \times \{3\}$ which is a hyperbolic metric space.

For the universal cover \tilde{B} of the thin block B , define:

$$\Pi_{\lambda,B}(x) = \Pi_{Bi}(x), x \in \tilde{S} \times \{i\}, i = 0, \dots, 3$$

$\Pi_{\lambda,B}$ is a retract - Thick Block

The proof for a thick block is exactly as in [28] and [30]. We omit it here.

Lemma 4.1 *(Lemma 4.1 of [30]) There exists $C > 0$ such that the following holds:*

Let $x, y \in \tilde{S} \times \{0, 1\} \subset \tilde{B}$ for some thick block B . Then $d(\Pi_{\lambda,B}(x), \Pi_{\lambda,B}(y)) \leq Cd(x, y)$.

$\Pi_{\lambda,B}$ is a retract - Amalgamation Block

The main ingredient in this case is Lemma 3.10.

Lemma 4.2 *There exists $C > 0$ such that the following holds:*

Let $x, y \in \tilde{S} \times \{0, 1, 2, 3\} \subset \tilde{B}$ for some amalgamated block B . Then $d_{Gel}(\Pi_{\lambda,B}(x), \Pi_{\lambda,B}(y)) \leq Cd_{Gel}(x, y)$.

Proof: It is enough to show this for the following cases:

1. $x, y \in \tilde{S} \times \{0\}$ OR $x, y \in \tilde{S} \times \{3\}$.
2. $x = (p, 0)$ and $y = (p, 1)$ for some p
3. x, y both lie in the geometric core K
4. $x = (p, 2)$ and $y = (p, 3)$ for some p .

Case 1: This follows from Lemma 3.8

Case 2 and Case 4: These follow from Lemma 3.10 which says that the hyperbolic and electric projections of p onto the hyperbolic geodesic

$[a, b]$ and the electro-ambient geodesic $[a, b]_{ea}$ respectively ‘almost agree’. If π_h and π_e denote the hyperbolic and electric projections, then there exists $C_1 > 0$ such that

$$d_{Gel}(\pi_h(p), \pi_e(p)) \leq C_1$$

Hence

$$d_{Gel}(\Pi_{\lambda, B}((p, i)), \Pi_{\lambda, B}((p, i + 1))) \leq C_1 + 1, \text{ for } i = 0, 2.$$

Case 3: This follows from the fact that K in the graph model with the electric metric is essentially the tree coming from the splitting. Further, by the properties of π_e , each amalgamation component projects down to a set of diameter zero. Hence

$$d_{Gel}(\Pi_{\lambda, B}(p), \Pi_{\lambda, B}(q)) \leq C_1 + 1$$

Choosing C as the maximum of these constants, we are through. \square

5 Construction of Quasiconvex Sets and Quasigeodesics

5.1 Construction of B_λ and Π_λ

Given a manifold M of amalgamated geometry, we know that M is homeomorphic to $S \times J$ for $J = [0, \infty)$ or $(-\infty, \infty)$. By definition of amalgamated geometry, there exists a sequence I_i of intervals and blocks B_i where the metric on $S \times I_i$ coincides with that on some building block B_i . Denote:

- $B_{\mu, B_i} = B_{i\mu}$
- $\Pi_{\mu, B_i} = \Pi_{i\mu}$

Now for a block $B = S \times I$ (thick or amalgamated), a natural map Φ_B may be defined taking $\mu = B_{\mu, B} \cap \tilde{S} \times \{0\}$ to a geodesic $B_{\mu, B} \cap \tilde{S} \times \{k\} = \Phi_B(\mu)$ where $k = 1$ or 3 according as B is thick or amalgamated. Let the map Φ_{B_i} be denoted as Φ_i for $i \geq 0$. For $i < 0$ we shall modify this by defining $\tilde{\Phi}_i$ to be the map that takes $\mu = B_{\mu, B_i} \cap \tilde{S} \times \{k\}$ to a geodesic $B_{\mu, B_i} \cap \tilde{S} \times \{0\} = \tilde{\Phi}_i(\mu)$ where $k = 1$ or 3 according as B is thick or amalgamated.

We start with a reference block B_0 and a reference geodesic segment $\lambda = \lambda_0$ on the ‘lower surface’ $\tilde{S} \times \{0\}$. Now inductively define:

- $\lambda_{i+1} = \Phi_i(\lambda_i)$ for $i \geq 0$
- $\lambda_{i-1} = \Phi_i(\lambda_i)$ for $i \leq 0$
- $B_{i\lambda} = B_{\lambda_i}(B_i)$
- $\Pi_{i\lambda} = \Pi_{\lambda_i, B_i}$
- $B_\lambda = \bigcup_i B_{i\lambda}$
- $\Pi_\lambda = \bigcup_i \Pi_{i\lambda}$

Recall that each $\tilde{S} \times i$ for $i = 0 \cdots m$ is called a **horizontal sheet** of \tilde{B} , where $m = 1$ or 3 according as B is thick or amalgamated. We will restrict our attention to the union of the horizontal sheets \widetilde{M}_H of \widetilde{M} with the metric induced from the graph model.

Clearly, $B_\lambda \subset \widetilde{M}_H \subset \widetilde{M}$, and Π_λ is defined from \widetilde{M}_H to B_λ . Since \widetilde{M}_H is a ‘coarse net’ in \widetilde{M} (equipped with the *graph model metric*), we will be able to get all the coarse information we need by restricting ourselves to \widetilde{M}_H .

By Lemmas 4.1 and 4.2, we obtain the fact that each $\Pi_{i\lambda}$ is a retract. Hence assembling all these retracts together, we have the following basic theorem:

Theorem 5.1 *There exists $C > 0$ such that for any geodesic $\lambda = \lambda_0 \subset \tilde{S} \times \{0\} \subset \tilde{B}_0$, the retraction $\Pi_\lambda : \widetilde{M}_H \rightarrow B_\lambda$ satisfies:*

$$\text{Then } d_{Gel}(\Pi_{\lambda, B}(x), \Pi_{\lambda, B}(y)) \leq C d_{Gel}(x, y) + C.$$

Note 1 For Theorem 5.1 above, note that all that we really require is that the universal cover \tilde{S} be a hyperbolic metric space. There is *no restriction on \widetilde{M}_H* . In fact, Theorem 5.1 would hold for general stacks of hyperbolic metric spaces with blocks of amalgamated geometry.

Note 2: M_H has been built up out of **graph models of thick and amalgamated blocks** and have sheets that are electrocuted along geodesics.

We want to make *Note 1* above explicit. We first modify the definition of amalgamation geometry as follows, retaining only local quasiconvexity.

Definition: A manifold M homeomorphic to $S \times J$, where $J = [0, \infty)$ or $J = (-\infty, \infty)$, is said to be a model of **weak amalgamation geometry** if

1. there is a fiber preserving homeomorphism from M to $\tilde{S} \times J$ that lifts to a quasi-isometry of universal covers

2. there exists a sequence I_i of intervals (with disjoint interiors) and blocks B_i where the metric on $S \times I_i$ is the same as that on some building block B_i . Each block is either thick or has amalgamation geometry.
3. $\bigcup_i I_i = J$
4. There exists $C_0 > 0$ such that for all amalgamated blocks B_i and geometric cores $C \subset B_i$, all amalgamation components of \tilde{C} are C_0 -quasiconvex in \tilde{B}_i

Then as a consequence of *the proof* of Theorem 5.1, we have the following Corollary.

Corollary 5.2 *Let M be a model manifold of weak amalgamation geometry. There exists $C > 0$ such that the following holds:*

Given any geodesic $\lambda \subset \tilde{S} \times \{0\}$, let B_λ, Π_λ be as before. Then for $\lambda = \lambda_0 \subset \tilde{S} \times \{0\} \subset \tilde{B}_0$, the retraction $\Pi_\lambda : \tilde{M}_H \rightarrow B_\lambda$ satisfies:

$$\text{Then } d_{\text{Gel}}(\Pi_{\lambda,B}(x), \Pi_{\lambda,B}(y)) \leq C d_{\text{Gel}}(x, y) + C.$$

In fact, all that follows in this section may just as well be done for model manifolds of *weak amalgamation geometry*. We shall make this explicit again at the end of this entire section.

But before we proceed, we would like to deduce one further Corollary of Theorem 5.1, which shall be useful towards the end of the paper. Instead of constructing vertical hyperbolic ladders B_λ for finite geodesic segments, first note that λ might as well be bi-infinite. Next, we would like to construct such a B_λ **equivariantly** under the action of \mathbb{Z} . That is to say, we would like to construct a vertical annulus in the manifold M homeomorphic to $S \times \mathbb{R}$.

To do this, we start with a simple closed geodesic σ on $S \times \{0\}$. Instead of performing the construction in the universal cover, homotop σ into $S \times \{i\}$ for each level i . Let σ_i denote the shortest electro-ambient geodesic in the *free homotopy class* of $\sigma \times \{i\}$ in the path pseudometric on $S \times \{i\}$. Now let B_σ denote the set $B_\sigma = \bigcup_i \tilde{\sigma}_i$. Then the proof of Theorem 5.1 ensures the quasiconvexity of B_σ in the *graph-metric*. Finally, since B_σ has been

constructed to be equivariant under the action of the surface group, its quotient in M is an embedded ‘quasi-annulus’ $A_{P\sigma}$ which partitions the manifold locally. We use the term ‘quasi-annulus’ because $A_{P\sigma}$ is a collection of disjoint circles at different levels. We finally conclude:

Corollary 5.3 *Let M be a model manifold of weak amalgamation geometry. There exists $C > 0$ such that the following holds:*

Given any simple closed geodesic $\sigma \subset S \times \{0\}$, let B_σ be as above. Then its quotient, the embedded quasi-annulus $A_{P\sigma}$ above is C -quasiconvex in M with the graph metric.

Another Corollary will be used later. Suppose $\Sigma = \Sigma \times \{0\}$ be a subsurface of $S \times \{0\}$ with geodesic boundary components $\sigma^1 \cdots \sigma^k$. Let Σ_i be the subsurface of $S \times \{i\}$ that is bounded by $\sigma_i^1 \cdots \sigma_i^k$. Let $B_\Sigma = \bigcup_i \Sigma_i$.

Corollary 5.4 *Let M be a model manifold of weak amalgamation geometry. There exists $C > 0$ such that the following holds:*

Given any subsurface $\Sigma \subset S \times \{0\}$ with geodesic boundary components, let B_Σ be as above. Then B_Σ is C -quasiconvex in M with the graph metric.

5.2 Heights of Blocks

Recall that each geometric core $C \subset B$ is identified with $S \times I$ where each fibre $\{x\} \times I$ has length $\leq l_C$ for some l_C , called the *thickness* of the block B . If $C \subset B_i$ for one of the above blocks B_i , we shall denote l_C as l_i .

Instead of considering all the horizontal sheets, we would now like to consider only the **boundary horizontal sheets**, i.e. for a thick block we consider $\tilde{S} \times \{0, 1\}$ and for a thin block we consider $\tilde{S} \times \{0, 3\}$. The union of all boundary horizontal sheets will be denoted by M_{BH} .

Observation 1: $\widetilde{M_{BH}}$ is a ‘coarse net’ in \widetilde{M} in the **graph model**, but not in the **model of amalgamated geometry**.

In the graph model, any point can be connected by a vertical segment of length ≤ 2 to one of the boundary horizontal sheets.

However, in the model of amalgamated geometry, there are points within amalgamation components which are at a distance of the order of l_i from the boundary horizontal sheets. Since l_i is arbitrary, $\widetilde{M_{BH}}$ is no longer a ‘coarse net’ in \widetilde{M} equipped with the model of *amalgamated geometry*.

Observation 2: $\widetilde{M_H}$ is defined only in the **graph model**, but not in the model of amalgamated geometry.

Observation 3: The electric metric on the model of amalgamated geometry on \widetilde{M} obtained by electrocuting amalgamation components is quasi-isometric to the graph model of \widetilde{M} .

Bounded Height of Thick Block

Let $\mu \subset \widetilde{S} \times \{0\} \widetilde{B}_i$ be a geodesic in a (thick or amalgamated) block. Then there exists a (K_i, ϵ_i) -quasi-isometry ψ_i ($= \phi_i$ for thick blocks) from $\widetilde{S} \times \{0\}$ to $\widetilde{S} \times \{1\}$ and Ψ_i is the induced map on geodesics. Hence, for any $x \in \mu$, $\psi_i(x)$ lies within some bounded distance C_i of $\Psi_i(\mu)$. But x is connected to $\psi_i(x)$ by

Case 1 - Thick Blocks: a vertical segment of length 1

Case 2 - Amalgamated Blocks: the union of

1. two vertical segments of length 1 between $\widetilde{S} \times \{i\}$ and $\widetilde{S} \times \{i+1\}$ for $i = 0, 2$
2. a horizontal segment of length bounded by (some uniform) C' (cf. Lemma 3.3) connecting $(x, 1)$ to a point on the electro-ambient geodesic $B_\lambda(B) \cap \widetilde{S} \times \{1\}$
3. a vertical segment of electric length zero in the **graph model** connecting $(x, 1)$ to $(x, 2)$. Such a path has to travel *through an amalgamated block* in the model of **amalgamated geometry** and has length less than l_i , where l_i is the thickness of the i th block B_i .
4. a horizontal segment of length less than C' (Lemma 3.3) connecting $(\phi_i(x), 3)$ to a point on the hyperbolic geodesic $B_\lambda(B) \cap \widetilde{S} \times \{3\}$

Thus x can be connected to a point on $x' \in \Psi_i(\mu)$ by a path of length less than $g(i) = 2 + 2C' + l_i$. Recall that λ_i is the geodesic on the lower horizontal surface of the block \widetilde{B}_i . The same can be done for blocks \widetilde{B}_{i-1} and *going down* from λ_i to λ_{i-1} . What we have thus shown is:

Lemma 5.5 *There exists a function $g : \mathbb{Z} \rightarrow \mathbb{N}$ such that for any block B_i (resp. B_{i-1}), and $x \in \lambda_i$, there exists $x' \in \lambda_{i+1}$ (resp. λ_{i-1}) for $i \geq 0$ (resp. $i \leq 0$), satisfying:*

$$d(x, x') \leq g(i)$$

5.3 Admissible Paths

We want to define a collection of B_λ -**elementary admissible paths** lying in a bounded neighborhood of B_λ . B_λ is not connected. Hence, it does not make much sense to speak of the path-metric on B_λ . To remedy this we introduce a ‘thickening’ (cf. [16]) of B_λ which is path-connected and where the paths are controlled. A B_λ -**admissible path** will be a composition of B_λ -elementary admissible paths.

Recall that admissible paths in the graph model of bounded geometry consist of the following :

1. Horizontal segments along some $\tilde{S} \times \{i\}$ for $i = \{0, 1, 2, 3\}$ (amalgamated blocks) or $i = \{0, 1\}$ (thick blocks).
2. Vertical segments $x \times [0, 1]$ or $x \times [2, 3]$ for amalgamated blocks, where $x \in \tilde{S}$.
3. Hyperbolic geodesic segments of length $\leq l_B$ in $K \subset B$ joining $x \times \{1\}$ to $x \times \{2\}$ for amalgamated blocks.
4. Vertical segments of length 1 joining $x \times \{0\}$ to $x \times \{1\}$ for thick blocks.

We shall choose a subclass of these admissible paths to define B_λ -elementary admissible paths.

B_λ -elementary admissible paths in the thick block

Let $B = S \times [i, i + 1]$ be a thick block, where each (x, i) is connected by a vertical segment of length 1 to $(x, i + 1)$. Let ϕ be the map that takes (x, i) to $(x, i + 1)$. Also Φ is the map on geodesics induced by ϕ . Let $B_\lambda \cap \tilde{B} = \lambda_i \cup \lambda_{i+1}$ where λ_i lies on $\tilde{S} \times \{i\}$ and λ_{i+1} lies on $\tilde{S} \times \{i + 1\}$. π_j , for $j = i, i + 1$ denote nearest-point projections of $\tilde{S} \times \{j\}$ onto λ_j . Next, since ϕ is a quasi-isometry, there exists $C > 0$ such that for all $(x, i) \in \lambda_i$, $(x, i + 1)$ lies in a C -neighborhood of $\Phi(\lambda_i) = \lambda_{i+1}$. The same holds for ϕ^{-1} and points in λ_{i+1} , where ϕ^{-1} denotes the *quasi-isometric inverse* of ϕ from $\tilde{S} \times \{i + 1\}$ to $\tilde{S} \times \{i\}$. The B_λ -**elementary admissible paths** in \tilde{B} consist of the following:

1. Horizontal geodesic subsegments of λ_j , $j = \{i, i + 1\}$.

2. Vertical segments of length 1 joining $x \times \{0\}$ to $x \times \{1\}$.
3. Horizontal geodesic segments lying in a C -neighborhood of λ_j , $j = i, i+1$.

B_λ -elementary admissible paths in the amalgamated block

Let $B = S \times [i, i+3]$ be an amalgamated block, where each $(x, i+1)$ is connected by a geodesic segment of zero electric length and hyperbolic length $\leq C(B)$ (due to bounded thickness of B) to $(\phi(x), i+2)$ (Here ϕ can be thought of as the map from $\tilde{S} \times \{i+1\}$ to $\tilde{S} \times \{i+2\}$ that is the identity on the first component. Also Φ is the map on canonical representatives of electric geodesics induced by ϕ . Let $B_\lambda \cap \tilde{B} = \bigcup_{j=i \dots i+3} \lambda_j$ where λ_j lies on $\tilde{S} \times \{j\}$. π_j denotes nearest-point projection of $\tilde{S} \times \{j\}$ onto λ_j (in the appropriate sense - hyperbolic for $j = i, i+3$ and electric for $j = i+1, i+2$). Next, since ϕ is an electric isometry, but a hyperbolic quasi-isometry, there exists $C > 0$ (uniform constant) and $K = K(B)$ such that for all $(x, i) \in \lambda_i$, $(\phi(x), i+1)$ lies in an (electric) C -neighborhood and a hyperbolic K -neighborhood of $\Phi(\lambda_{i+1}) = \lambda_{i+2}$. The same holds for ϕ^{-1} and points in λ_{i+2} , where ϕ^{-1} denotes the *quasi-isometric inverse* of ϕ from $\tilde{S} \times \{i+2\}$ to $\tilde{S} \times \{i+1\}$.

Again, since λ_{i+1} and λ_{i+2} are electro-ambient quasigeodesics, we further note that there exists $C > 0$ (assuming the same C for convenience) such that for all $(x, i) \in \lambda_i$, $(x, i+1)$ lies in a (hyperbolic) C -neighborhood of λ_{i+1} . Similarly for all $(x, i+2) \in \lambda_{i+2}$, $(x, i+3)$ lies in a (hyperbolic) C -neighborhood of λ_{i+3} . The same holds if we go ‘down’ from λ_{i+1} to λ_i or from λ_{i+3} to λ_{i+2} . The B_λ -elementary admissible paths in \tilde{B} consist of the following:

1. Horizontal subsegments of λ_j , $j = \{i, \dots i+3\}$.
2. Vertical segments of length 1 joining $x \times \{j\}$ to $x \times \{j+1\}$, for $j = i, i+2$.
3. Horizontal geodesic segments lying in a *hyperbolic* C -neighborhood of λ_j , $j = i, \dots i+3$.
4. Horizontal hyperbolic segments of *electric length* $\leq C$ and *hyperbolic length* $\leq K(B)$ joining points of the form $(\phi(x), i+2)$ to a point on λ_{i+2} for $(x, i+1) \in \lambda_{i+1}$.
5. Horizontal hyperbolic segments of *electric length* $\leq C$ and *hyperbolic length* $\leq K(B)$ joining points of the form $(\phi^{-1}(x), i+1)$ to a point on λ_{i+1} for $(x, i+2) \in \lambda_{i+2}$.

Definition: A B_λ -admissible path is a union of B_λ -elementary admissible paths.

The following lemma follows from the above definition and Lemma 5.5.

Lemma 5.6 *There exists a function $g : \mathbb{Z} \rightarrow \mathbb{N}$ such that for any block B_i , and x lying on a B_λ -admissible path in \widetilde{B}_i , there exist $y \in \lambda_j$ and $z \in \lambda_k$ where $\lambda_j \subset B_\lambda$ and $\lambda_k \subset B_\lambda$ lie on the two boundary horizontal sheets, satisfying:*

$$\begin{aligned} d(x, y) &\leq g(i) \\ d(x, z) &\leq g(i) \end{aligned}$$

Let $h(i) = \sum_{j=0 \dots i} g(j)$ be the sum of the values of $g(j)$ as j ranges from 0 to i (with the assumption that increments are by $+1$ for $i \geq 0$ and by -1 for $i \leq 0$). Then we have from Lemma 5.6 above,

Corollary 5.7 *There exists a function $h : \mathbb{Z} \rightarrow \mathbb{N}$ such that for any block B_i , and x lying on a B_λ -admissible path in \widetilde{B}_i , there exist $y \in \lambda_0 = \lambda$ such that:*

$$d(x, y) \leq h(i)$$

Important Note: In the above Lemma 5.6 and Corollary 5.7, it is important to note that the distance d is **hyperbolic**, not electric. This is because the number l_i occurring in elementary paths of type 5 and 6 is a hyperbolic length depending only on i (in B_i).

Next suppose that λ lies outside $B_N(p)$, the N -ball about a fixed reference point p on the boundary horizontal surface $\widetilde{S} \times \{0\} \subset \widetilde{B}_0$. Then by Corollary 5.7, any x lying on a B_λ -admissible path in \widetilde{B}_i satisfies

$$d(x, p) \geq N - h(i)$$

Also, since the electric, and hence hyperbolic ‘thickness’ (the shortest distance between its boundary horizontal sheets) is ≥ 1 , we get,

$$d(x, p) \geq |i|$$

Assume for convenience that $i \geq 0$ (a similar argument works, reversing signs for $i < 0$). Then,

$$d(x, p) \geq \min\{i, N - h(i)\}$$

Let $h_1(i) = h(i) + i$. Then h_1 is a monotonically increasing function on the integers. If $h_1^{-1}(N)$ denote the largest positive integer n such that $h(n) \leq m$, then clearly, $h_1^{-1}(N) \rightarrow \infty$ as $N \rightarrow \infty$. We have thus shown:

Lemma 5.8 *There exists a function $M(N) : \mathbb{N} \rightarrow \mathbb{N}$ such that $M(N) \rightarrow \infty$ as $N \rightarrow \infty$ for which the following holds:*

For any geodesic $\lambda \subset \tilde{S} \times \{0\} \subset \tilde{B}_0$, a fixed reference point $p \in \tilde{S} \times \{0\} \subset \tilde{B}_0$ and any x on a B_λ -admissible path,

$$d(\lambda, p) \geq N \Rightarrow d(x, p) \geq M(N).$$

As pointed out before, the discussion and Lemmas of the previous two subsections go through just as well in the context of *weak amalgamation geometry* manifolds. We make this explicit in the case of Lemma 5.8 above.

Corollary 5.9 *Let M be a model manifold of **weak amalgamation geometry**. Then there exists a function $M(N) : \mathbb{N} \rightarrow \mathbb{N}$ such that $M(N) \rightarrow \infty$ as $N \rightarrow \infty$ for which the following holds:*

Given any geodesic $\lambda \subset \tilde{S} \times \{0\}$, let B_λ be as before. For $\lambda \subset \tilde{S} \times \{0\} \subset \tilde{B}_0$, a fixed reference point $p \in \tilde{S} \times \{0\} \subset \tilde{B}_0$ and any x on a B_λ -admissible path,

$$d(\lambda, p) \geq N \Rightarrow d(x, p) \geq M(N).$$

5.4 Joining the Dots

Recall that **admissible paths** in a model manifold of bounded geometry consist of:

1. Horizontal segments along some $\tilde{S} \times \{i\}$ for $i = \{0, 1, 2, 3\}$ (thin blocks) or $i = \{0, 1\}$ (thick blocks).
2. Vertical segments $x \times [0, 1]$ or $x \times [2, 3]$ for amalgamated blocks.
3. Vertical segments of length $\leq l_i$ joining $x \times \{1\}$ to $x \times \{2\}$ for amalgamated blocks.

4. Vertical segments of length 1 joining $x \times \{0\}$ to $x \times \{1\}$ for thick blocks.

Our strategy in this subsection is:

- 1 Start with an electric geodesic β_e in $\widetilde{M_{Gel}}$ joining the end-points of λ .
- 2 Replace it by an *admissible quasigeodesic*, i.e. an admissible path that is a quasigeodesic.
- 3 Project the intersection of the admissible quasigeodesic with the horizontal sheets onto B_λ .
- 4 The result of step 3 above is disconnected. *Join the dots* using B_λ -admissible paths.

The end product is an electric quasigeodesic built up of B_λ admissible paths.

Now for the first two steps:

- Since \widetilde{B} (for a thick block B) has thickness 1, any path lying in a thick block can be perturbed to an admissible path lying in \widetilde{B} , changing the length by at most a bounded multiplicative factor.
- For B amalgamated, we decompose paths into horizontal paths lying in some $\widetilde{S} \times \{j\}$, for $j = 0, \dots, 3$ and vertical paths of types (2) or (3) above. This can be done without altering electric length within $\widetilde{S} \times [1, 2]$. To see this, project any path \overline{ab} beginning and ending on $\widetilde{S} \times \{1, 2\}$ onto $\widetilde{S} \times \{1\}$ along the fibres. To connect this to the starting and ending points a, b , we have to at most adjoin vertical segments through a, b . Note that this does not increase the electric length of \overline{ab} , as the electric length is determined by the number of amalgamation blocks that \overline{ab} traverses.
- For paths lying in $\widetilde{S} \times [0, 1]$ or $\widetilde{S} \times [2, 3]$, we can modify the path into an admissible path, changing lengths by a bounded multiplicative constant. The result is therefore an electric quasigeodesic.
- Without loss of generality, we can assume that the electric quasigeodesic is one without back-tracking (as this can be done without increasing the length of the geodesic - see [12] or [19] for instance).
- Abusing notation slightly, assume therefore that β_e is an admissible electric quasigeodesic without backtracking joining the end-points of λ .

This completes Steps •1 and •2.

- Now act on $\beta_e \cap \widetilde{M_H}$ by Π_λ . From Theorem 5.1, we conclude, by restricting Π_λ to the horizontal sheets of $\widetilde{M_{Gel}}$ that the image $\Pi_\lambda(\beta_e)$ is a ‘dotted electric quasigeodesic’ lying entirely on B_λ . This completes step 3.
- Note that since β_e consists of admissible segments, we can arrange so that

two nearest points on $\beta_e \cap \widetilde{M}_H$ which are not connected to each other form the end-points of a vertical segment of type (2), (3) or (4). Let $\Pi_\lambda(\beta_e) \cap B_\lambda = \beta_d$, be the dotted quasigeodesic lying on B_λ . We want to join the dots in β_d converting it into a **connected** electric quasigeodesic built up of B_λ -**admissible paths**.

- For vertical segments of type (4) joining p, q (say), $\Pi_\lambda(p), \Pi_\lambda(q)$ are a bounded hyperbolic distance apart. Hence, by the proof of Lemma 4.1, we can join $\Pi_\lambda(p), \Pi_\lambda(q)$ by a B_λ -admissible path of length bounded by some C_0 (independent of B, λ).
- For vertical segments of type (2) joining p, q , we note that $\Pi_\lambda(p), \Pi_\lambda(q)$ are a bounded hyperbolic distance apart. Hence, by the proof of Lemma 4.2, we can join $\Pi_\lambda(p), \Pi_\lambda(q)$ by a B_λ -admissible path of length bounded by some C_1 (independent of B, λ).
- This leaves us to deal with case (3). Such a segment consists of a segment lying within a lift of an amalgamation block. Such a piece has electric length one in the graph model. Its image, too, has electric length one (See for instance, Case (3) of the proof of Lemma 4.2, where we noted that the projection of any amalgamation component lies within an amalgamation component).

After joining the dots, we can assume further that the quasigeodesic thus obtained does not backtrack (cf [12] and [19]).

Putting all this together, we conclude:

Lemma 5.10 *There exists a function $M(N) : \mathbb{N} \rightarrow \mathbb{N}$ such that $M(N) \rightarrow \infty$ as $N \rightarrow \infty$ for which the following holds:*

For any geodesic $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B}_0$, and a fixed reference point $p \in \widetilde{S} \times \{0\} \subset \widetilde{B}_0$, there exists a connected electric quasigeodesic β_{adm} without backtracking, such that

- β_{adm} is built up of B_λ -admissible paths.
- β_{adm} joins the end-points of λ .
- $d(\lambda, p) \geq N \Rightarrow d(\beta_{adm}, p) \geq M(N)$.

Proof: The first two criteria follow from the discussion preceding this lemma. The last follows from Lemma 5.8 since the discussion above gives a quasigeodesic built up out of admissible paths. \square

As in the previous subsections, Lemma 5.10 goes through for **weak amalgamation geometry**. We state this below:

Corollary 5.11 *Suppose that M is a manifold of **weak amalgamation geometry**. There exists a function $M(N) : \mathbb{N} \rightarrow \mathbb{N}$ such that $M(N) \rightarrow \infty$ as $N \rightarrow \infty$ for which the following holds:*

For any geodesic $\lambda \subset \tilde{S} \times \{0\} \subset \tilde{B}_0$, and a fixed reference point $p \in \tilde{S} \times \{0\} \subset \tilde{B}_0$, there exists a connected electric quasigeodesic β_{adm} without backtracking, such that

- β_{adm} is built up of B_λ -admissible paths.
- β_{adm} joins the end-points of λ .
- $d(\lambda, p) \geq N \Rightarrow d(\beta_{adm}, p) \geq M(N)$.

5.5 Admissible Quasigeodesics and Electro-ambient Quasigeodesics

Definition: We next define (as before) a (k, ϵ) electro-ambient quasigeodesic γ in \tilde{M} relative to the amalgamation components \tilde{K} to be a (k, ϵ) quasigeodesic in the graph model of \tilde{M} such that in an ordering (from the left) of the amalgamation components that γ meets, each $\gamma \cap \tilde{K}$ is a (k, ϵ) - quasigeodesic in the induced path-metric on \tilde{K} .

This subsection is devoted to extracting an electro-ambient quasigeodesic β_{ea} from a B_λ -admissible quasigeodesic β_{adm} . β_{ea} shall satisfy the property indicated by Lemma 5.10 above. We shall prove this Lemma under the assumption of (strong) amalgamation geometry. However, a weaker assumption (which we shall discuss later, while weakening *amalgamation geometry* to **graph amalgamation geometry**) is enough for the main Lemma of this subsection to go through.

Lemma 5.12 *There exist κ, ϵ and a function $M'(N) : \mathbb{N} \rightarrow \mathbb{N}$ such that $M'(N) \rightarrow \infty$ as $N \rightarrow \infty$ for which the following holds:*

For any geodesic $\lambda \subset \tilde{S} \times \{0\} \subset \tilde{B}_0$, and a fixed reference point $p \in \tilde{S} \times \{0\} \subset \tilde{B}_0$, there exists a (κ, ϵ) electro-ambient quasigeodesic β_{ea} without backtracking, such that

- β_{ea} joins the end-points of λ .
- $d(\lambda, p) \geq N \Rightarrow d(\beta_{ea}, p) \geq M'(N)$.

Proof: From Lemma 5.10, we have a B_λ - admissible quasigeodesic β_{adm} and a function $M(N)$ without backtracking satisfying the conclusions of the Lemma. Since β_{adm} does not backtrack, we can decompose it as a union of

non-overlapping segments β_1, \dots, β_k , such that each β_i is either an admissible (hyperbolic) quasigeodesic lying outside amalgamation components, or a B_λ -admissible quasigeodesic lying entirely within some amalgamation component \widetilde{K}_i . Further, since β_{adm} does not backtrack, we can assume that all K_i 's are distinct.

We modify β_{adm} to an electro-ambient quasigeodesic β_{ea} as follows:

1. β_{ea} coincides with β_{adm} outside amalgamation components.
2. There exist κ, ϵ such that if some β_i lies within an amalgamation component \widetilde{K}_i then, by uniform quasiconvexity of the K_i 's, it may be replaced by a (κ, ϵ) (hyperbolic) quasigeodesic β_i^{ea} joining the end-points of β_i and lying within \widetilde{K}_i .

The resultant path β_{ea} is clearly an electro-ambient quasigeodesic without backtracking. Next, each component β_i^{ea} lies in a C_i neighborhood of β_i , where C_i depends only on the thickness l_i of the amalgamation component K_i .

We let $C(n)$ denote the maximum of the values of C_i for $K_i \subset B_n$. Then, as in the proof of Lemma 5.8, we have for any $z \in \beta_{ea} \cap B_n$,

$$d(z, p) \geq \max(n, M(N) - C(n))$$

Again, as in Lemma 5.8, this gives us a (new) function $M'(N) : \mathbb{N} \rightarrow \mathbb{N}$ such that $M'(N) \rightarrow \infty$ as $N \rightarrow \infty$ for which

- $d(\lambda, p) \geq N \Rightarrow d(\beta_{ea}, p) \geq M'(N)$.

This prove the Lemma. \square

Note: We have essentially used the following two properties of amalgamation components in concluding Lemma 5.12:

1. any path lying inside an amalgamation component \widetilde{K} may be replaced by a (uniform) hyperbolic quasigeodesic joining its end-points and lying within the same \widetilde{K}
2. Each electro-ambient quasigeodesic joining the end-points of an admissible quasigodesic in $\widetilde{K} \subset B_n$ lies in a (hyperbolic) $C(n)$ -neighborhood of the latter.

We shall have occasion to use this when we discuss **graph-quasiconvexity**.

6 Cannon-Thurston Maps for Surfaces Without Punctures

It is now time to introduce hyperbolicity of \widetilde{M} , global quasiconvexity of amalgamation components, (and hence) model manifolds of (strong) amalgamation geometry. We shall assume till the end of this section that

1. there exists a hyperbolic manifold M and a homeomorphism from \widetilde{M} to $\widetilde{S} \times \mathbb{R}$. We identify \widetilde{M} with $\widetilde{S} \times \mathbb{R}$ via this homeomorphism.
2. $\widetilde{S} \times \mathbb{R}$ admits a quasi-isometry g to a model manifold of *amalgamated geometry*
3. g preserves the fibers over $\mathbb{Z} \subset \mathbb{R}$

We shall henceforth ignore the quasi-isometry g and think of \widetilde{M} itself as the universal cover of a model manifold of *amalgamated geometry*.

6.1 Electric Geometry Revisited

We note the following properties of the pair (X, \mathcal{H}) where X is the graph model of \widetilde{M} and \mathcal{H} consists of the amalgamation components. There exist C, D, Δ such that

1. Each amalgamation component is C -quasiconvex.
2. Any two amalgamation components are 1-separated.
3. $\widetilde{M}_{Gel} = X_{Gel}$ is Δ -hyperbolic, (where $\widetilde{M}_{Gel} = X_{Gel}$ is the electric metric on $\widetilde{M} = X$ obtained by electrocuting all amalgamation components, i.e. all members of \mathcal{H}).
4. Given K, ϵ , there exists D_0 such that if γ be a (K, ϵ) hyperbolic quasigeodesic joining a, b and if β be a (K, ϵ) electro-ambient quasigeodesic joining a, b , then γ lies in a D_0 neighborhood of β .

The first property follows from the definition of a manifold of amalgamation geometry.

The second follows from the construction of the graph model.

The third follows from Lemma 3.2.

The fourth follows from Lemma 3.5.

6.2 Proof of Theorem

We shall now assemble the proof of the main Theorem.

Theorem 6.1 *Let M be a 3 manifold homeomorphic to $S \times J$ (for $J = [0, \infty)$ or $(-\infty, \infty)$). Further suppose that M has amalgamated geometry, where $S_0 \subset B_0$ is the lower horizontal surface of the building block B_0 . Then the inclusion $i : \tilde{S} \rightarrow \tilde{M}$ extends continuously to a map $\hat{i} : \hat{S} \rightarrow \hat{M}$. Hence the limit set of \tilde{S} is locally connected.*

Proof: Suppose $\lambda \subset \tilde{S}$ lies outside a large N -ball about p . By Lemma 5.12 we obtain an electro-ambient quasigeodesic without backtracking β_{ea} lying outside an $M(N)$ -ball about p (where $M(N) \rightarrow \infty$ as $N \rightarrow \infty$).

Suppose that β_{ea} is a (κ, ϵ) electro-ambient quasigeodesic. Note that κ, ϵ depend on ‘the Lipschitz constant’ of Π_λ and hence only on \tilde{S} and \tilde{M} .

From Property (4) above, (or Lemma 3.5) we find that if β^h denote the hyperbolic geodesic in \tilde{M} joining the end-points of λ , then β^h lies in a (uniform) C' neighborhood of β_{ea} .

Let $M_1(N) = M(N) - C'$. Then $M_1(N) \rightarrow \infty$ as $N \rightarrow \infty$. Further, the hyperbolic geodesic β^h lies outside an $M_1(N)$ -ball around p . Hence, by Lemma 2.1, the inclusion $i : \tilde{S} \rightarrow \tilde{M}$ extends continuously to a map $\hat{i} : \hat{S} \rightarrow \hat{M}$.

Since the continuous image of a compact locally connected set is locally connected (see [17]) and the (intrinsic) boundary of \tilde{S} is a circle, we conclude that the limit set of \tilde{S} is locally connected.

This proves the theorem. \square

7 Modifications for Surfaces with Punctures

In this section, we shall describe the modifications necessary for Theorem 6.1 to go through for surfaces with punctures.

7.1 Partial Electrocution

In this subsection, we indicate a modification of Farb’s [12] notion of *strong relative hyperbolicity* and construction of an electric metric, described earlier in this paper. Though much of this works in the context of relative hyperbolicity with *Bounded Penetration Property* [12] or, equivalently, strong relative hyperbolicity [2], we shall focus on the case we need, viz. convex hyperbolic 3-manifolds with boundary of the form $\sigma \times P$, where P is either an interval or

a circle, and σ is a horocycle of some fixed length e_0 . In the universal cover, if we excise (open) horoballs, we are left with a manifold whose boundaries are flat horospheres of the form $\tilde{\sigma} \times \tilde{P}$. Note that $\tilde{P} = P$ if P is an interval, and \mathbb{R} if P is a circle (the case for a $(Z + Z)$ -cusp).

Partial Electrocution of a horosphere H will be defined as putting the zero metric in the $\tilde{\sigma}$ direction, and retaining the usual Euclidean metric in the \tilde{P} direction.

The construction of *partially electrocuted* horospheres is half way between the spirit of Farb's construction (in Lemmas 3.2, 3.11, where the entire horosphere is coned off), and McMullen's Theorem 3.13 (where nothing is coned off, and properties of *ambient quasigeodesics* are investigated).

In the partially electrocuted case, instead of coning all of a horosphere down to a point we cone only horocyclic leaves of a foliation of the horosphere. Effectively, therefore, we have a cone-line rather a cone-point.

We explicitly describe below *partial electrocution* for convex hyperbolic 3-manifolds.

Partial Electrocution of Horospheres

Let Y be a convex simply connected hyperbolic 3-manifold. Let \mathcal{B} denote a collection of horoballs. Let X denote Y minus the interior of the horoballs in \mathcal{B} . Let \mathcal{H} denote the collection of boundary horospheres. Then each $H \in \mathcal{H}$ with the induced metric is isometric to a Euclidean product $E^{n-2} \times L$ for an interval $L \subset \mathbb{R}$. Partially electrocute each H by giving it the product of the zero metric with the Euclidean metric, i.e. on E^{n-2} give the zero metric and on L give the Euclidean metric. The resulting space is exactly what one would get by gluing to each H the mapping cylinder of the projection of H onto the L -factor.

Much of what follows would go through in the following more general setting:

1. X is (strongly) hyperbolic relative to a collection of subsets H_α , thought of as horospheres (and *not horoballs*).
2. For each H_α there is a uniform large-scale retraction $g_\alpha : H_\alpha \rightarrow L_\alpha$ to some (uniformly) δ -hyperbolic metric space L_α , i.e. there exist $\delta, K, \epsilon > 0$ such that for all H_α there exists a δ -hyperbolic L_α and a map $g_\alpha : H_\alpha \rightarrow L_\alpha$ with $d_{L_\alpha}(g_\alpha(x), g_\alpha(y)) \leq K d_{H_\alpha}(x, y) + \epsilon$ for all $x, y \in H_\alpha$.

3. The coned off space corresponding to H_α is the (metric) mapping cylinder for the map $g_\alpha : H_\alpha \rightarrow L_\alpha$.

In Farb's construction L_α is just a single point. However, the notions and arguments of [12] or Klarreich [19] or the proof of quasiconvexity of a *hyperbolic geodesic union horoballs it meets* in McMullen [21] go through even in this setting. The metric, and geodesics and quasigeodesics in the partially electrocuted space will be referred to as the partially electrocuted metric d_{pel} , and partially electrocuted geodesics and quasigeodesics respectively. In this situation, we conclude as in Lemma 3.2:

Lemma 7.1 *(X, d_{pel}) is a hyperbolic metric space and the sets L_α are uniformly quasiconvex.*

Note 1: When K_α is a point, the last statement is a triviality.

Note 2: (X, d_{pel}) is strongly hyperbolic relative to the sets $\{L_\alpha\}$. In fact the space obtained by electrocuting the sets L_α in (X, d_{pel}) is just the space (X, d_e) obtained by electrocuting the sets $\{H_\alpha\}$ in X .

Note 3: The proof of Lemma 7.1 and other such results below follow Farb's [12] constructions. For instance, consider a hyperbolic geodesic η in a convex complete simply connected hyperbolic 3-manifold X . Let H_i , $i = 1 \cdots k$ be the partially electrocuted horoballs it meets. Let $N(\eta)$ denote the union of η and H_i 's. Let Y denote X minus the interiors of the H_i 's. The first step is to show that $N(\eta) \cap Y$ is quasiconvex in (Y, d_{pel}) . To do this one takes a hyperbolic R -neighborhood of $N(\eta)$ and projects (Y, d_{pel}) onto it, using the hyperbolic projection. It was shown by Farb in [12] that the projections of all horoballs are uniformly bounded in hyperbolic diameter. (This is essentially mutual coboundedness). Hence, given K , choosing R large enough, any path that goes out of an R -neighborhood of $N(\eta)$ cannot be a K -partially electrocuted quasigeodesic. This is the one crucial step that allows the results of [12], in particular, Lemma 7.1 to go through in the context of partially electrocuted spaces.

As in Lemma 3.11, partially electrocuted quasigeodesics and geodesics without backtracking have the same intersection patterns with *horospheres and boundaries of lifts of tubes* as electric geodesics without backtracking. Further, since electric geodesics and hyperbolic quasigeodesics have similar intersection patterns with *horoballs and lifts of tubes* it follows that partially

electrocuted quasigeodesics and hyperbolic quasigeodesics have similar intersection patterns with *horospheres and boundaries of lifts of tubes*. We state this formally below:

Lemma 7.2 *Given $K, \epsilon \geq 0$, there exists $C > 0$ such that the following holds:*

Let γ_{pel} and γ denote respectively a (K, ϵ) partially electrocuted quasigeodesic in (X, d_{pel}) and a hyperbolic (K, ϵ) -quasigeodesic in (Y, d) joining a, b . Then $\gamma \cap X$ lies in a (hyperbolic) C -neighborhood of (any representative of) γ_{pel} . Further, outside of a C -neighborhood of the horoballs that γ meets, γ and γ_{pel} track each other.

Next, we note that partial electrocution preserves quasiconvexity. Suppose that $A \subset Y$ as also $A \cap H$ for all $H \in \mathcal{H}$ are C -quasiconvex. Then given $a, b \in A \cap X$, the hyperbolic geodesic λ in X joining a, b lies in a C -neighborhood of A . Since horoballs are convex, λ cannot backtrack. Let λ_{pel} be the partially electrocuted geodesic joining $a, b \in (X, d_{pel})$. Then by Lemma repel-track above, we conclude that for all $H \in \mathcal{H}$ that λ intersects, there exist points of λ_{pel} (hyperbolically) near the entry and exit points of λ with respect to H . Since these points lie near $A \cap H$, and since the corresponding L is quasiconvex in (X, d_{pel}) , we conclude that λ_{pel} lies within a bounded distance from A near horoballs. For the rest of λ_{pel} the conclusion follows from Lemma 7.2. We conclude:

Lemma 7.3 *Given C_0 there exists C_1 such that if $A \subset Y$ and $A \cap H$ are C_0 -quasiconvex for all $H \in \mathcal{H}$, then (A, d_{pel}) is C_1 -quasiconvex in (X, d_{pel}) .*

7.2 Amalgamated Geometry for Surfaces with Punctures

Step 1: For a hyperbolic surface S^h (possibly) with punctures, we fix a (small) e_0 , and excise the cusps leaving horocyclic boundary components of (ordinary or Euclidean) length e_0 . We then take the induced *path metric* on S^h minus cusps and call the resulting surface S . This induced path metric will still be referred to as the hyperbolic metric on S (with the understanding that now S possibly has boundary).

Step 2: The definitions and constructions of **amalgamated building blocks** and **amalgamation components** now go through with appropriate changes. The only difference is that S now might have boundary curves of length e_0 . For thick blocks, we assume (as in [30]) that a thick block is

the universal curve over a Teichmüller geodesic (of length less than D for some uniform D) minus $\text{cusps} \times I$.

There is one subtle point about global quasiconvexity (in \tilde{M}) of amalgamation components. This does not hold in the metric obtained by merely excising the cusps and equipping the resulting horospheres with the Euclidean metric. What we demand is that each amalgamation component along with the parts of the horoballs that meet the boundary (horocycle times closed interval)'s be quasiconvex in \tilde{M} . When we partially electrocute horospheres below, and consider quasiconvexity in the resulting partially electrocuted space, amalgamation components in this sense remain quasiconvex by Lemma 7.3.

Step 3: Next, we modify the metric on S by electrocuting its boundary components so that the metric on the boundary components of each block $S \times I$ is the product of the zero metric on the horocycles of fixed (Euclidean) length e_0 and the Euclidean metric on the I -factor. The resulting blocks will be called **partially electrocuted blocks**. We demand that in the model M_{pel} obtained by gluing together partially electrocuted blocks, the amalgamation components are uniformly quasiconvex. By Lemma 7.3, this follows from quasiconvexity of amalgamation components in the sense of the note above. Note that M_{pel} may also be constructed directly from M by excising a neighborhood of the cusps and partially electrocuting the resulting horospheres. By Lemma 7.1 \tilde{M}_{pel} is a hyperbolic metric space.

Step 4: Again, the definitions and constructions of **amalgamated building blocks** and **amalgamation components** now go through *mutatis mutandis* for partially electrocuted blocks.

Step 5: Next, let λ^h be a hyperbolic geodesic in \tilde{S}^h . We replace pieces of λ^h that lie within horodisks by shortest horocyclic segments joining its entry and exit points (into the corresponding horodisk). Such a path is called a horo-ambient quasigeodesic in [29]. See Figure below:

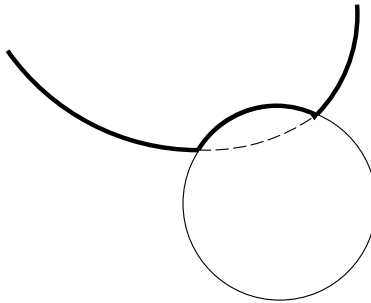


Figure 4: *Horo-ambient quasigeodesic*

A small modification might be introduced if we electrocute horocycles. Geodesics and quasigeodesics without backtracking then travel for free along the zero metric horocycles. This does not change matters much as the geodesics and quasigeodesics in the two resulting constructions track each other by Lemma 3.11.

Step 6: Thus, our starting point for the construction of the hyperbolic ladder B_λ is not a hyperbolic geodesic λ^h but a horoambient quasigeodesic λ . We construct the **graph model** as before. *By Lemma 7.3 quasiconvexity of amalgamation components as well as lifts of Margulis tubes is preserved by partial electrocution.*

Step 7: The construction of B_λ, Π_λ and their properties go through *mutatis mutandis* and we conclude that B_λ is quasiconvex in the *graph model* of the partially electrocuted space \tilde{M}_{pel} . As before, $\widetilde{M}_{H_{pel}}$ will denote the collection of horizontal sheets. The modification of Theorem 5.1 is given below:

Theorem 7.4 *There exists $C > 0$ such that for any horo-ambient geodesic $\lambda = \lambda_0 \subset \tilde{S} \times \{0\} \subset \widetilde{B}_0$, the retraction $\Pi_\lambda : \widetilde{M}_{H_{pel}} \rightarrow B_\lambda$ satisfies:*

$$d_{pel}(\Pi_{\lambda,B}(x), \Pi_{\lambda,B}(y)) \leq Cd(x, y) + C.$$

Step 8: From this step on, the modifications for punctured surfaces follow [29] As in [29], we decompose λ into portions λ^c and λ^b that lie along horocycles and those that do not. Accordingly, we decompose B_λ into two parts B_λ^c and B_λ^b consisting of parts that lie along horocycles and those that do not. Dotted geodesics and admissible paths are constructed as before. As in Lemma 5.8, we get

Lemma 7.5 *There exists a function $M(N) : \mathbb{N} \rightarrow \mathbb{N}$ such that $M(N) \rightarrow \infty$ as $N \rightarrow \infty$ for which the following holds:
For any horo-ambient quasigeodesic $\lambda \subset \tilde{S} \times \{0\} \subset \widetilde{B}_0$, a fixed reference point $p \in \tilde{S} \times \{0\} \subset \widetilde{B}_0$ and any x on B_λ^b ,*

$$d(\lambda^b, p) \geq N \Rightarrow d(x, p) \geq M(N).$$

Step 9: Construct a ‘dotted’ ambient electric quasigeodesic lying on B_λ by projecting some(any) ambient electric quasigeodesic onto B_λ by Π_λ . Join the

dots using admissible paths to get a connected ambient electric quasigeodesic β_{amb} .

Step 10 Construct from $\beta_{amb} \subset \widetilde{M}$ an electric quasigeodesic γ in \widetilde{M}_{pel} as in the previous section and note that parts of γ not lying along horocycles lie close to B_λ^b .

Step 11 Conclude that if λ^h lies outside large balls in S^h then each point of γ lying outside partially electrocuted horospheres also lies outside large balls.

Step 12 Let γ^h denote the hyperbolic geodesic in \widetilde{M}^h joining the end-points of γ . By Lemma 7.2 γ and γ^h track each other off a bounded (hyperbolic) neighborhood of the electrocuted horoballs. Recall that X denotes \widetilde{M}^h minus interiors of horoballs. Then, every point of $\gamma^h \cap X$ must lie close to some point of γ lying outside partially electrocuted horospheres. Hence from Step (11), if λ^h lies outside large balls about p in S^h then $\gamma^h \cap X$ also lies outside large balls about p in X . In particular, γ^h enters and leaves horoballs at large distances from p . From this we conclude that γ^h lies outside large balls. Hence by Lemma 2.1 there exists a Cannon-Thurston map and the limit set is locally connected.

We state the conclusion below:

Theorem 7.6 *Let M^h be a 3 manifold homeomorphic to $S^h \times J$ (for $J = [0, \infty)$ or $(-\infty, \infty)$). Further suppose that M^h has amalgamated geometry, where $S_0^h \subset B_0$ is the lower horizontal surface of the building block B_0 . Then the inclusion $i : \tilde{S}^h \rightarrow \widetilde{M}^h$ extends continuously to a map $\hat{i} : \hat{S}^h \rightarrow \widehat{M}^h$. Hence the limit set of \tilde{S}^h is locally connected.*

8 Weakening the Hypothesis I: Graph Quasiconvexity and Graph Amalgamation Geometry

We now proceed to weaken the hypothesis of amalgamation geometry in the hope of capturing all Kleinian surface groups. Recall that in the definition of amalgamation geometry, two criteria were used - *local and global quasiconvexity of amalgamation components*. We shall retain local quasiconvexity, and replace global quasiconvexity by a **weaker condition** which we shall term **graph quasiconvexity**. The rationale behind this terminology shall be made clear later. We first modify the definition of amalgamation geometry as follows, retaining only local quasiconvexity. We first recall the

definition of *weak amalgamation geometry*.

A manifold M homeomorphic to $S \times J$, where $J = [0, \infty)$ or $J = (-\infty, \infty)$, is said to be a model of **weak amalgamation geometry** if

1. there is a fiber preserving homeomorphism from M to $\tilde{S} \times J$ that lifts to a quasi-isometry of universal covers
2. there exists a sequence I_i of intervals (with disjoint interiors) and blocks B_i where the metric on $S \times I_i$ is the same as that on some building block B_i . Each block is either thick or has amalgamation geometry.
3. $\bigcup_i I_i = J$
4. There exists $C > 0$ such that for all amalgamated blocks B_i and geometric cores $K \subset B_i$, all amalgamation components of \tilde{K} are C -quasiconvex in \tilde{B}_i

Definition: An amalgamation component $K \subset B_n$ is said to be (m, κ) **graph - quasiconvex** if there exists a κ -quasiconvex (in the hyperbolic metric) subset $CH(K)$ containing K such that

1. $CH(K) \subset N_m^G(K)$ where $N_m^G(K)$ denotes the m neighborhood of K in the graph model of M .
2. For each K there exists C_K such that K is C_K -quasiconvex in $CH(K)$.

Since the quasiconvex sets (thought of as convex hulls of K) lie within a bounded distance from K in the *graph model* we have used the term *graph-quasiconvex*.

Definition: A manifold M of weak amalgamation geometry is said to be a model of **graph amalgamation geometry** if there exist m, κ such that each amalgamation geometry component is (m, κ) -graph - quasiconvex.

A manifold N is said to have **graph amalgamation geometry** if there is a level-preserving homeomorphism from N to a model manifold of *graph*

amalgamation geometry that lifts to a quasi-isometry at the level of universal covers.

Note: As before, we proceed with the assumption that for surfaces with punctures, S corresponds to a complete hyperbolic surface S^h minus a neighborhood of the cusps with horocycles electrocuted. Further, M corresponds to M^h minus a neighborhood of the cusps with resultant horospheres *partially electrocuted*.

Now, let us indicate the modifications necessary to carry out the proof of the Cannon-Thurston Property for manifolds of graph amalgamation geometry (suppressing the quasi-isometry to a model manifold). As in Theorem 6.1, the proof consists of two steps:

1. Constructing a quasiconvex set B_λ in an auxiliary electric space (the **graph model**), and from this an admissible electric quasigeodesic β .
2. Recovering from β and its intersection pattern, information about the hyperbolic geodesic joining its end-points.

The first step is the same as that for models of *amalgamation geometry* as it goes through for *weak amalgamation geometry*. Then from Corollary 5.2 we have:

Step 1A: Given $\lambda \subset \tilde{S} \times \{0\}$, construct B_λ, Π_λ as before. There exists $C > 0$ such that the retraction $\Pi_\lambda : \widetilde{M_H} \rightarrow B_\lambda$ satisfies:
 $d_{Gel}(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C d_{Gel}(x, y) + C$, where d_{Gel} denotes the metric in the graph model.

Again, from Corollary 5.11 we have:

Step 1B:

There exists a function $M(N) : \mathbb{N} \rightarrow \mathbb{N}$ such that $M(N) \rightarrow \infty$ as $N \rightarrow \infty$ for which the following holds:

For any geodesic $\lambda \subset \tilde{S} \times \{0\} \subset \widetilde{B_0}$, and a fixed reference point $p \in \tilde{S} \times \{0\} \subset \widetilde{B_0}$, there exists a connected B_λ -admissible quasigeodesic β_{adm} without backtracking, such that

- β_{adm} is built up of B_λ -admissible paths.
- β_{adm} joins the end-points of λ .
- $d(\lambda, p) \geq N \Rightarrow d(\beta_{adm}, p) \geq M(N)$. (d is the ordinary, non-electric metric.)

Summary of Step 2:

Now we come to the second step: **recovering a hyperbolic geodesic from an electric geodesic.**

This step can be further subdivided into two parts. In the first part we construct a second auxiliary space M_2 by electrocuting the elements $CH(K)$. We show that the spaces \tilde{M}_1 and \tilde{M}_2 are quasi-isometric. In fact we show that the identity map on the underlying subset is a quasi-isometry. This step requires only the first condition in the definition of *graph quasiconvexity*. The second stage extracts information about an electro-ambient quasi-geodesic in \tilde{M}_2 from an admissible path in \tilde{M}_1 . It is at this second stage that we require the second condition: (not necessarily uniform) quasi-convexity of amalgamation components.

We now furnish the details.

Step 2A:

Let M_1 denote M with the graph metric obtained by electrocuting amalgamation components. Next, let M_2 denote M with an electric metric obtained by electrocuting the family of sets $CH(K)$ (for amalgamation components K) appearing in the definition of **graph amalgamation geometry**.

Lemma 8.1 *The identity map on the underlying set M from M_1 to M_2 induces a quasi-isometry of universal covers \tilde{M}_1 and \tilde{M}_2 .*

Proof: Let d_1, d_2 denote the electric metrics on \tilde{M}_1 and \tilde{M}_2 . Since $K \subset CH(K)$ for every amalgamation component, we have right off

$$d_1(x, y) \leq d_2(x, y) \text{ for all } x, y \in \tilde{M}$$

To prove a reverse inequality with appropriate constants, it is enough to show that each set $CH(K)$ (of diameter one in M_2) has uniformly bounded diameter in M_1 . To see this, note that by definition of graph-quasiconvexity, there exists n such that for all K and each point a in $CH(K)$, there exists a point $b \in K$ with $d_1(x, y) \leq n$. Hence by the triangle inequality,

$$d_2(x, y) \leq 2n + 1 \text{ for all } x, y \in \widetilde{CH(K)}$$

Therefore,

$$d_2(x, y) \leq (2n + 1)d_1(x, y) \text{ for all } x, y \in \tilde{M}$$

This proves the Lemma. \square

Step 2B:

Now let β_{adm} denote an admissible B_λ quasigeodesic in \tilde{M}_1 , which does not backtrack relative to the amalgamation components. By Lemma 8.1 above, β_{adm} is a quasigeodesic in \tilde{M}_2 . As in Lemma 5.12, using the Note following it, we conclude:

There exists a κ, ϵ -electro-ambient quasigeodesic β_{ea} in \tilde{M}_2 (as opposed to \tilde{M}_1 , which is what we needed in the *amalgamation geometry* case). (See Lemma 5.12.) Note that in \tilde{M}_2 , we electrocute the lifts of the sets $CH(K)$ rather than \tilde{K} 's.

We thus obtain, as in Lemma 5.12 a function $M'(N) : \mathbb{N} \rightarrow \mathbb{N}$ such that $M'(N) \rightarrow \infty$ as $N \rightarrow \infty$ for which the following holds:

For any geodesic $\lambda \subset \tilde{S} \times \{0\} \subset \tilde{B}_0$, and a fixed reference point $p \in \tilde{S} \times \{0\} \subset \tilde{B}_0$, there exists a (κ, ϵ) electro-ambient quasigeodesic β_{ea} without backtracking, such that

- β_{ea} joins the end-points of λ .
- $d(\lambda, p) \geq N \Rightarrow d(\beta_{ea}, p) \geq M'(N)$.

Finally, as in the proof of Theorem 6.1, we use Lemma 3.5 to conclude that the hyperbolic geodesic in \tilde{M} joining the end-points of λ lies in a uniform hyperbolic neighborhood of β_{ea} . This gives us Theorem 6.1 with **graph amalgamation geometry** replacing *amalgamation geometry*.

Theorem 8.2 *Let M be a 3 manifold homeomorphic to $S \times J$ (for $J = [0, \infty)$ or $(-\infty, \infty)$). Further suppose that M has graph amalgamation geometry, where $S_0 \subset B_0$ is the lower horizontal surface of the building block B_0 . Then the inclusion $i : \tilde{S} \rightarrow \tilde{M}$ extends continuously to a map $\hat{i} : \hat{\tilde{S}} \rightarrow \hat{\tilde{M}}$. Hence the limit set of \tilde{S} is locally connected.*

The modifications for the case with punctures are as before (See Theorem 7.6. Thus, we conclude:

Theorem 8.3 *Let M^h be a 3 manifold homeomorphic to $S^h \times J$ (for $J = [0, \infty)$ or $(-\infty, \infty)$). Further suppose that M^h has amalgamated geometry, where $S_0^h \subset B_0$ is the lower horizontal surface of the building block B_0 . Then the inclusion $i : \tilde{S}^h \rightarrow \tilde{M}^h$ extends continuously to a map $\hat{i} : \hat{\tilde{S}}^h \rightarrow \hat{\tilde{M}}^h$. Hence the limit set of \tilde{S}^h is locally connected.*

9 Weakening the Hypothesis II: Split Geometry

In this section, we shall weaken the hypothesis of *graph amalgamation geometry* further to include the possibility of Margulis tubes cutting across the blocks B_i . But before we do this, let us indicate a straightforward generalisation of *amalgamation geometry* or *graph amalgamation geometry*

9.1 More Margulis Tubes in a Block

A straightforward generalisation of Theorem 6.1 (or Theorem 8.2) is to the case where more than one Margulis tube is allowed per block B , and each of these tubes splits the block B locally. On the surface S , this corresponds to a number of disjoint (uniformly) bounded length curves. As before we require that each amalgamation component be uniformly quasiconvex (or graph quasiconvex) in \tilde{M} for the proof of Theorem 6.1 (or Theorem 8.2) to go through. See the figure below for a schematic rendering of the model block of amalgamation geometry.

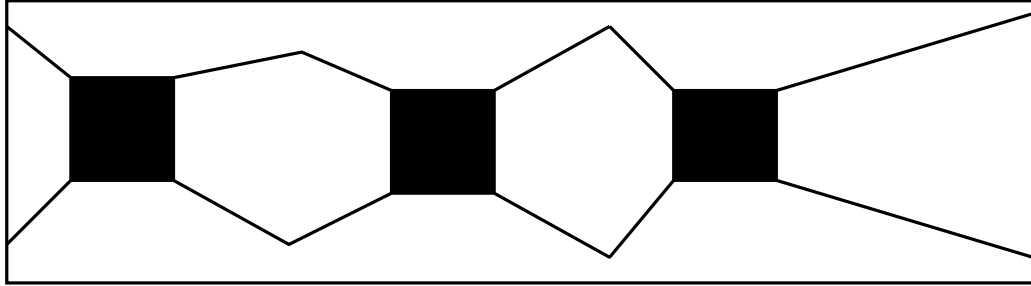


Figure 5: Building Block for Generalised Amalgamation Geometry

9.2 Motivation for Split Geometry

So far, we have assumed that the boundaries of *amalgamated geometry blocks* or *graph amalgamated geometry blocks* are all of bounded geometry. This assumption needs to be relaxed to accomodate general surface Kleinian groups. Before we define the objects of interest, we shall first informally analyse what went into the construction of the hyperbolic ladder B_λ . We require:

1. Horizontal surfaces S_i , all abstractly homeomorphic to each other
2. A block decomposition $M = \cup B_i$, where $B_{i-1} \cap B_i = S_i$
3. Given a geodesic $\lambda_i \subset \tilde{S}_i$, we require a (uniformly) large-scale retract π_i of \tilde{S}_i onto λ_i and a prescription to construct $\lambda_{i+1} \subset \tilde{S}_{i+1}$. Thus, starting with $\lambda_0 \subset \tilde{S}_0$, we first construct π_0 and then inductively construct the pairs (λ_i, π_i) .
4. Each block B_i has an auxiliary metric or pseudometric which induces the given path metrics on S_{i-1}, S_i .

We want to relax the assumption that S_i 's have bounded geometry, while retaining the essential properties of bounded geometry. As elsewhere in this paper we invoke the following (uncomfortably dictatorial) policy that we have adopted:

Policy: *Electrocute anything that gives trouble.*

What this policy means is that whenever some construction possibly gives rise to non-uniformity of some parameter(s), locate the source of non-uniformity and electrocute it. Then, at the end of the game, re-instate the original geometry by using comparison properties between ordinary hyperbolic geometry and electric geometry.

Thus, each S_i is now allowed to have a pseudometric where a finite number of disjoint, bounded length (uniformly, independent of i) collection of simple closed geodesics are electrocuted. Then, instead of geodesics $\lambda_i \subset \tilde{S}_i$, we shall require the λ_i to be only electro-ambient geodesics. This will allow us to go ahead with the construction of B_λ .

One further comment as to how this solves the problem. Let us fix a small (less than Margulis constant) ϵ_0 . Given any hyperbolic surface S^h , we can simply electrocute *thin parts*, i.e. tubular neighborhoods of short (less than ϵ_0) geodesics with boundaries of length ϵ_0 . Alternately, we can first cut out the interiors of these thin parts. Next, corresponding to each *Margulis annulus* that has been cut out, glue the corresponding boundary components of length ϵ_0 together, and then electrocute the resulting closed curves.

This construction is adapted to the construction of *split level surfaces* in Minsky [25], and Brock-Canary-Minsky [8].

9.3 Definitions

Topologically, a **split subsurface** S^s of a surface S is a (possibly disconnected, proper) subsurface with boundary such that $S - S^s$ consists of a non-empty family of non-homotopic annuli, which in turn are not homotopic into the boundary of S^s .

Geometrically, we assume that S is given some finite volume hyperbolic structure. A split subsurface S^s of S has bounded geometry, i.e.

1. each boundary component of S^s is of length ϵ_0 , and is in fact a component of the boundary of $N_k(\gamma)$, where γ is a hyperbolic geodesic on S , and $N_k(\gamma)$ denotes its k -neighborhood.
2. For any closed geodesic β on S , either $\beta \subset S - S^s$, or, the length of any component of $\beta \cap (S - S^s)$ is greater than ϵ_0 .

Topologically, a **split block** $B^s \subset B = S \times I$ is a topological product $S^s \times I$ for some *connected* S^s . However, its upper and lower boundaries need not be $S^s \times 1$ and $S^s \times 0$. We only require that the upper and lower boundaries be split subsurfaces of S^s . This is to allow for Margulis tubes starting (or ending) within the split block. Such tubes would split one of the horizontal boundaries but not both. We shall call such tubes **hanging tubes**. See figure below:

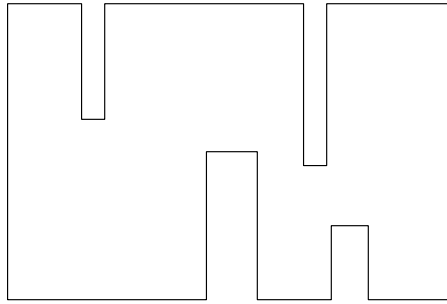


Figure 6: *Split Block with hanging tubes*

Geometrically, we require that the metric on a split block induces a path metric on its *upper and lower horizontal* boundary components, which are

subsurfaces of $S^s \times \partial I$, such that each horizontal boundary component is a (geometric) split surface. Further, the metric on B^s induces on each *vertical* boundary component of a Margulis tube $\partial S^s \times I$ the product metric. Each boundary component for Margulis tubes that ‘travel all the way from the lower to the upper boundary’ is an annulus of height equal to length of I . We demand further that *hanging tubes* have *length uniformly bounded below by* $\eta_0 > 0$. Further, each such annulus has cross section a round circle of length ϵ_0 . This leaves us to decide the metric on lower and upper boundaries of hanging tubes. Such boundaries are declared to have a metric equal to that on $S^1 \times [-\eta, \eta]$, where S^1 is a round circle of length ϵ_0 and η is a sufficiently small number.

Note: In the above definition, we *do not* require that the upper (or lower) horizontal boundary of a split block B^s be connected for a connected B^s . This happens due to the presence of *hanging tubes*.

We further require that the distance between horizontal boundary components is at least 1, i.e. for a component R of S^s $d(R \times 0, R \times 1) \geq 1$. We define the **thickness** of a split block to be the supremum of the lengths of $x \times I$ for $x \in S^s$ and demand that it be finite (which holds under all reasonable conditions, e.g. a smooth metric; however, since we shall have occasion to deal with possibly discontinuous pseudometrics, we make this explicit). We shall denote the thickness of a split block B^s by l_B .

Each component of a split block shall be called a **split component**. We further require that the ‘vertical boundaries’ (corresponding to Euclidean annuli) of split components be uniformly (independent of choice of a block and a split component) quasiconvex in the corresponding split component.

Note that the boundary of each split block has an intrinsic metric that is flat and corresponds to a Euclidean torus.

A lift of a split block to the universal cover of the block $B = S \times I$ shall be termed a **split component** of \tilde{B} .

Remark: The notion of *split components* we deal with here is closely related to the notion of **bands** described by Bowditch in [5], [6] and also to the notion of **scaffolds** introduced by Brock, Canary and Minsky in [8].

We define a **welded split block** to be a split block with identifications as follows: Components of $\partial S^s \times 0$ are glued together if and only if they correspond to the same geodesic in $S - S^s$. The same is done for components of $\partial S^s \times 1$. A simple closed curve that results from such an identification shall be called a **weld curve**. For hanging tubes, we also weld the boundary circles of their *lower or upper boundaries* by simply collapsing $S^1 \times [-\eta, \eta]$

to $S^1 \times \{0\}$.

This may be done topologically or geometrically while retaining Dehn twist information about the curves. To record information about the Dehn twists, we have to define (topologically) a map that takes the lower boundary of a welded split block to the upper boundary. We define a map that takes $x \times 0$ to $x \times 1$ for every point in S^s . This clearly induces a map from the lower boundary of a welded split block to its upper boundary. However, this is not enough to give a well-defined map on paths. To do this, we have to record *twist information* about *weld curves*. The way to do this is to define a map on transversals to weld curves. The map is defined on transversals by recording the number of times a transversal to a weld curve $\gamma \times 0$ twists around $\gamma \times 1$ on the upper boundary of the welded split block. (A related context in which such transversal information is important is that of markings described in Minsky [25].)

Let the metric product $S^1 \times [0, 1]$ be called the **standard annulus** if each horizontal S^1 has length ϵ_0 . For hanging tubes the standard annulus will be taken to be $S^1 \times [0, 1/2]$.

Next, we require another pseudometric on B which we shall term the **tube-electrocuted metric**. We first define a map from each boundary annulus $S^1 \times I$ (or $S^1 \times [0, 1/2]$ for hanging annulii) to the corresponding standard annulus that is affine on the second factor and an isometry on the first. Now glue the mapping cylinder of this map to the boundary component. The resulting ‘split block’ has a number of standard annulii as its boundary components. Call the split block B^s with the above mapping cylinders attached, the *stabilized split block* B^{st} .

Glue boundary components of B^{st} corresponding to the same geodesic together to get the **tube electrocuted metric** on B as follows. Suppose that two boundary components of B^{st} correspond to the same geodesic γ . In this case, these boundary components are both of the form $S^1 \times I$ or $S^1 \times [0, \frac{1}{2}]$ where there is a projection onto the horizontal S^1 factor corresponding to γ . Let $S_l^1 \times J$ and $S_r^1 \times J$ denote these two boundary components (where J denotes I or $[0, \frac{1}{2}]$). Then each $S^1 \times \{x\}$ has length ϵ_0 . Glue $S_l^1 \times J$ to $S_r^1 \times J$ by the natural ‘identity map’. Finally, on each resulting $S^1 \times \{x\}$ put the zero metric. Thus the annulus $S^1 \times J$ obtained via this identification has the zero metric in the *horizontal direction* $S^1 \times \{x\}$ and the Euclidean metric in the *vertical direction* J . The resulting block will be called the **tube-electrocuted block** B_{tel} and the pseudometric on it will be denoted as d_{tel} . Note that B_{tel} is homeomorphic to $S \times I$. The operation of obtaining

a *tube electrocuted block and metric* (B_{tel}, d_{tel}) from a split block B^s shall be called *tube electrocution*.

Next, fix a hyperbolic structure on a Riemann surface S and construct the metric product $S \times \mathbb{R}$. Fix further a positive real number l_0 .

Definition 9.1 *An annulus A will be said to be **vertical** if it is of the form $\sigma \times J$ for σ a geodesic of length less than l_0 on S and $J = [a, b]$ a closed sub-interval of \mathbb{R} . J will be called the **vertical interval** for the vertical annulus A .*

*A disjoint collection of annulii is said to be a **vertical system** of annulii if each annulus in the collection is vertical.*

The above definition is based on a definition due to Bowditch [5],[6].

Suppose now that $S \times \mathbb{R}$ is equipped with a vertical system \mathcal{A} of annulii. We shall call $z \in \mathbb{R}$ a

1. a **beginning level** if z is the lower bound of a vertical interval for some annulus $A \in \mathcal{A}$.
2. an **ending level** if z is the lower bound of a vertical interval for some annulus $A \in \mathcal{A}$.
3. an **intermediate level** if z is an interior point of a vertical interval for some annulus $A \in \mathcal{A}$.

In the figure below (where for convenience, all appropriate levels are marked with integers), 2, 5, 11 and 14 are *beginning levels*, 4, 7, 13 and 16 are *ending levels*, 3, 6, 9, 12 and 15 are *intermediate levels*. We shall also allow Dehn twists to occur while going along the annulus.

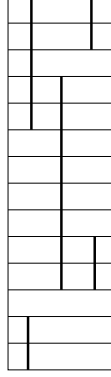


Figure 7: *Vertical Annulus Structure*

A slight modification of the vertical annulus structure will sometimes be useful.

Replacing each geodesic γ on S by a neighborhood $N_\epsilon(\gamma)$ for sufficiently small ϵ , we obtain a **vertical Margulis tube** structure after taking products with vertical intervals. The family of Margulis tubes shall be denoted by \mathcal{T} and the union of their interiors as $Int\mathcal{T}$. The union of $Int\mathcal{T}$ and its horizontal boundaries (corresponding to neighborhoods of geodesics $\gamma \subset S$) shall be denoted as $Int^+\mathcal{T}$.

Thick Block

Fix constants D, ϵ and let $\mu = [p, q]$ be an ϵ -thick Teichmüller geodesic of length less than D . μ is ϵ -thick means that for any $x \in \mu$ and any closed geodesic η in the hyperbolic surface S_x over x , the length of η is greater than ϵ . Now let B denote the universal curve over μ reparametrized such that the length of μ is covered in unit time. Thus $B = S \times [0, 1]$ topologically.

B is given the path metric and is called a **thick building block**.

Note that after acting by an element of the mapping class group, we might as well assume that μ lies in some given compact region of Teichmüller space. This is because the marking on $S \times \{0\}$ is not important, but rather its position relative to $S \times \{1\}$. Further, since we shall be constructing models only upto quasi-isometry, we might as well assume that $S \times \{0\}$ and $S \times \{1\}$ lie in the orbit under the mapping class group of some fixed base surface. Hence μ can be further simplified to be a Teichmüller geodesic joining a pair (p, q) amongst a finite set of points in the orbit of a fixed hyperbolic surface S .

Weak Split Geometry

A manifold $S \times \mathbb{R}$ equipped with a vertical Margulis tube structure is said to be a model of **weak split geometry**, if it is equipped with a new metric satisfying the following conditions:

1. $S \times [m, m+1] \cap \text{Int}\mathcal{T} = \emptyset$ (for $m \in \mathbb{Z} \subset \mathbb{R}$) implies that $S \times [m, m+1]$ is a thick block.
2. $S \times [m, m+1] \cap \text{Int}\mathcal{T} \neq \emptyset$ (for $m \in \mathbb{Z} \subset \mathbb{R}$) implies that $S \times [m, m+1] - \text{Int}^+\mathcal{T}$ is (geometrically) a split block.
3. There exists a uniform upper bound on the lengths of vertical intervals for vertical Margulis tubes
4. The metric on each component Margulis tube T of \mathcal{T} is hyperbolic

Note 1: Dehn twist information can still be implicitly recorded in a model of *weak split geometry* by the Dehn filling information corresponding to tubes T .

Note 2: The metric on a model of *weak split geometry* is possibly discontinuous along the boundary torii of Margulis tubes. If necessary, one could smooth this out. But we would like to carry on with the above metric.

Removing the interiors of Margulis tubes and tube electrocuting each block, we obtain a new pseudo-metric on M called the **tube electrocuted metric** d_{tel} on M . The pseudometric d_{tel} may also be lifted to \tilde{M} .

The induced pseudometric on \tilde{S}_i 's shall be referred to as **split electric metrics**. The notions of *electro-ambient metrics*, *geodesics* and *quasi-geodesics* go through in this context.

Next, we shall describe a **graph metric** on \tilde{M} which is almost (but not quite) the metric on the nerve of the covering of \tilde{M} by split components (where each edge is assigned length 1). This is not strictly true as thick blocks are retained with their usual geometry in the graph metric. However the analogy with the nerve is exact if all blocks have *weak split geometry*.

For each split component \tilde{K} assign a single vertex v_K and construct a cone of height $1/2$ with base \tilde{K} and vertex v_K . The metric on the resulting

space (coned-off or electric space in the sense of Farb [12]) shall be called the **graph metric** on \tilde{M} .

The union of a split component of \tilde{B} and the lifts of Margulis tubes (to \tilde{M}) that intersect its boundary shall be called a **split amalgamation component** in \tilde{M} .

Definition: A split amalgamation component K is said to be **(m. κ) - graph quasiconvex** if there exists a κ -quasiconvex (in the hyperbolic metric) subset $CH(K)$ containing K such that

1. $CH(K) \subset N_m^G(K)$ where $N_m^G(K)$ denotes the m neighborhood of K in the graph metric on M .
2. For each K there exists C_K such that K is C_K -quasiconvex in $CH(K)$.

Definition: A model manifold M of weak split geometry is said to be a model of **split geometry** if there exist m, κ such that each split amalgamation component is (m, κ) - graph quasiconvex.

9.4 The Cannon-Thurston Property for Manifolds of Split Geometry

We shall first extract information about geodesics in the *tube electrocuted* model. As with Theorem 6.1 and Theorem 8.2, the proof splits into two parts:

1. Construction of B_λ and its quasiconvexity in an auxiliary graph metric. The end-product of this step is an electro-ambient quasigeodesic in the graph model
2. Extraction of information about a hyperbolic geodesic and its intersection pattern with blocks from the electro-ambient quasigeodesic constructed in Step 1 above.

Details of Step 1:

Step 1A: Construction of B_λ

It is at this stage that the construction differs somewhat from the construction of B_λ for manifolds of *graph amalgamated geometry*.

We start with the (tube-electrocutted) metric d_{tel} on the model manifold of *split geometry*. Then there exists a sequence of split surfaces S_i exiting the end(s).

Recall that in the construction of B_λ (for all preceding cases) we are *not* interested in the metric on each \tilde{S}_i per se, but in geodesics on \tilde{S}_i .

The metric d_{tel} on the model manifold induces the **split electric metric** on each S_i obtained by electrocuting the **weld curves**. The natural geodesics to consider on \tilde{S}_i are therefore the *electro-ambient* quasigeodesics where the electrocutted subsets correspond to geodesics representing the weld curves.

Thus we start off with a hyperbolic geodesic λ in \tilde{S}_0 joining a, b say. We let λ_0 denote the electro-ambient quasigeodesic joining a, b in the split electric metric on \tilde{S}_0 . Now construct B_λ inductively as follows:

- Each split block B_i and hence \tilde{B}_i comes equipped with a (topological) product structure. Thus there is a canonical map $\Phi_i : \tilde{S}_i \rightarrow \tilde{S}_{i+1}$ which maps each (x, i) to a point $(x, i+1)$ by lifting the map from S_i to S_{i+1} ($i \geq 0$ corresponding to the product structure).
- Next, if λ_i is an electro-ambient quasi-geodesic in the split electric metric on \tilde{S}_i joining (a, i) and (b, i) we let λ_{i+1} denote the electro-ambient quasi-geodesic in the split-electric metric on \tilde{S}_{i+1} joining $(a, i+1)$ and $(b, i+1)$. This gives us a prescription for constructing λ_{i+1} from λ_i for $i \geq 0$. Similarly, for $i \leq 0$ (in the totally degenerate case) we can construct λ_{i-1} from λ_i . Then as before, define

$$B_\lambda = \bigcup_i \lambda_i$$

- Again, $\pi_i : \tilde{S}_i \rightarrow \lambda_i$ is defined as the retraction that minimises the ordered pair of distances in the split electric metric and the hyperbolic metric (without electrocutting weld curves). Π_λ is obtained in the graph metric by defining it on the horizontal sheets \tilde{S}_i as

$$\Pi_\lambda(x) = \pi_i(x) \text{ for } x \in \tilde{S}_i.$$

- Then as before we conclude that in the graph model for \tilde{M} , with the metric d_{Gel} , Π_λ does not stretch distances much, i.e. there exists a uniform $C \geq 0$ such that

$$d_{Gel}(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C d_{Gel}(x, y) + C$$

Step 1B: Construction of admissible quasigeodesic

The above construction of Π_λ may be used to construct a B_λ -admissible quasigeodesic β_{adm} in the tube-electrocuted model. As before we have:

There exists a function $M(N) : \mathbb{N} \rightarrow \mathbb{N}$ such that $M(N) \rightarrow \infty$ as $N \rightarrow \infty$ for which the following holds:

For any geodesic $\lambda \subset \tilde{S} \times \{0\} \subset \tilde{B}_0$, and a fixed reference point $p \in \tilde{S} \times \{0\} \subset \tilde{B}_0$, there exists a connected B_λ -admissible quasigeodesic β_{adm} without backtracking, such that

- β_{adm} is built up of B_λ -admissible paths.
- β_{adm} joins the end-points of λ .
- If $d(\lambda, p) \geq N$ then for any $x \in \beta_{adm} - \text{Int}\mathcal{T}$, $d(x, p) \geq M(N)$. (d is the ordinary, hyperbolic, or non-electric metric.)

Step 2: Recovering a quasigeodesic in the tube electrocuted model from an admissible quasigeodesic

We now follow the proof of Theorem 8.2.

Step 2A: As in *Step 2A* in the proof of Theorem 8.2 we construct a second auxiliary space M_2 by electrocuting the elements $CH(K)$ for split components K . The spaces \tilde{M}_1 and \tilde{M}_2 are quasi-isometric by uniform *graph quasiconvexity* of split components. In fact the identity map on the underlying subset is a quasi-isometry as in Lemma 8.1.

Step 2B Next, as in *Step 2B* in the proof of Theorem 8.2, we extract information about an electro-ambient quasi-geodesic in \tilde{M}_2 from an admissible path in \tilde{M}_1 . It is at this second stage that we require the condition that split components are (not necessarily uniformly) quasi-convex in the hyperbolic metric, and hence by Lemma 7.3 in the tube electrocuted metric d_{tel} .

We may assume that β_{adm} does not backtrack relative to the split components. From Step 2A above, β_{adm} is a quasigeodesic in \tilde{M}_2 . Then we conclude:

There exists a κ, ϵ -electro-ambient quasigeodesic β_{tea} in \tilde{M}_2 (Note that in \tilde{M}_2 , we electrocute the lifts of the sets $CH(K)$ rather than \tilde{K} 's).

We finally obtain a function $M'(N) : \mathbb{N} \rightarrow \mathbb{N}$ such that $M'(N) \rightarrow \infty$ as $N \rightarrow \infty$ for which the following holds:

For any geodesic $\lambda \subset \tilde{S} \times \{0\} \subset \tilde{B}_0$, and a fixed reference point $p \in \tilde{S} \times \{0\} \subset \tilde{B}_0$, there exists a (κ, ϵ) electro-ambient quasigeodesic β_{tea} (in the **tube electrocuted metric**) without backtracking, such that

- β_{tea} joins the end-points of λ .

- If λ lies outside a large ball about a fixed reference point $p \in \tilde{S}_0$, then each point of $\beta_{tea} \cap (\tilde{M} - \text{Int}\mathcal{T})$ also lies outside a large ball about p .

Step 3: Recovering a hyperbolic geodesic from the tube electrocuted quasigeodesic β_{tea}

This is a new step that comes from the extra phenomenon of tube electrocution which makes the metric d_{tel} an ‘intermediate’ metric between the hyperbolic metric d and the graph metric d_{Gel} .

Observe that lifts of Margulis tubes to (\tilde{M}, d_{Gel}) have uniformly bounded diameter in the metric d_{Gel} and consequently in the metric d_{tel} by uniform boundedness of vertical intervals of vertical Margulis tubes. Hence the *tube electrocuted metric* d_{tel} on \tilde{M} is quasi-isometric to the metric d_{fe} where lifts of Margulis tubes are electrocuted (i.e. fully electrocuted rather than just tube electrocuted, and hence each tube has diameter 1). Let \tilde{M}_{fe} denote \tilde{M} equipped with this new metric. Then geodesics without backtracking in the tube electrocuted metric become (uniform) quasi-geodesics without backtracking in \tilde{M}_{fe} .

Note: It is at this (rather late) stage that we need to assume that \tilde{M} is a hyperbolic metric space.

Let γ^h denote a hyperbolic geodesic joining the end-points of β_{tea} and hence λ . By Lemma 3.11, γ^h and β_{tea} track each other off Margulis tubes. Hence $\gamma^h \cap (\tilde{M} - \text{Int}\mathcal{T})$ lies outside a large ball about p . In particular, this is true for entry and exit points of γ^h with respect to Margulis tubes. This implies (See for instance Lemma 7.3 of [30]) that the parts of λ^h lying within Margulis tubes also lie outside large balls about p . As before, by Lemma 2.1 we infer the Cannon-Thurston property for manifolds of *split geometry*.

Theorem 9.2 *Let M be a 3 manifold homeomorphic to $S \times J$ (for $J = [0, \infty)$ or $(-\infty, \infty)$). Further suppose that M has split geometry, where $S_0 \subset B_0$ is the lower horizontal surface of the building block B_0 . Then the inclusion $i : \tilde{S} \rightarrow \tilde{M}$ extends continuously to a map $\hat{i} : \hat{S} \rightarrow \hat{M}$. Hence the limit set of \tilde{S} is locally connected.*

There is a bit of ineffective ambiguity in the above theorem. In split geometry, S_0 is only a split surface. We can extend this to any surface S_0 so long as the annuli that we glue on to construct the full surface lie entirely within Margulis tubes. The modifications for the case with punctures are as before: conclude:

Theorem 9.3 *Let M^h be a 3 manifold homeomorphic to $S^h \times J$ (for $J = [0, \infty)$ or $(-\infty, \infty)$). Further suppose that M^h has split geometry, where $S_0^h \subset B_0$ is the lower horizontal surface of the building block B_0 . Then the inclusion $i : \tilde{S}^h \rightarrow \tilde{M}^h$ extends continuously to a map $\hat{i} : \hat{S}^h \rightarrow \hat{M}^h$. Hence the limit set of \tilde{S}^h is locally connected.*

10 Generalisation: Incompressible away from Cusps

The aim of this section is to sketch the proof of the following more general theorem:

Theorem 10.1 *Suppose that $N^h \in H(M, P)$ is a hyperbolic structure of split geometry on a pared manifold (M, P) with incompressible boundary $\partial_0 M$. Let M_{gf} denote a geometrically finite hyperbolic structure adapted to (M, P) . Then the map $i : \tilde{M}_{gf} \rightarrow \tilde{N}^h$ extends continuously to the boundary $\hat{i} : \hat{M}_{gf} \rightarrow \hat{N}^h$. If Λ denotes the limit set of \tilde{M} , then Λ is locally connected.*

See [29] for definition of pared manifold with incompressible boundary (this coincides with the notion of manifolds whose boundary is incompressible away from cusps). Theorem 7.6 and its proof takes the place of Theorem 4.15 of [29]. Since nothing else is new, given these constituents, we content ourselves with giving an outline of the proof.

Outline of Proof of Theorem 10.1

Step 1 Construct B_λ in \tilde{M} ($= \tilde{M}^h$ - cusps) as in Section 4.1 of [29]. The only difference is that for an end E of *split geometry*, \tilde{E} is given the graph metric corresponding to the graph model.

Step 2 As in Sections 4.2, 4.3 of [29] we obtain a retract Π_λ onto B_λ .

Step 3 Construct a ‘dotted’ ambient electric quasigeodesic lying on B_λ by projecting some(any) (graph) geodesic onto B_λ by Π_λ .

Step 4 Join the dots using admissible paths. This results in a connected ambient electric quasigeodesic β_{amb} .

Step 4A Construct from β_{amb} an electro-ambient quasigeodesic β_{ea} replacing bits that lie within blocks by hyperbolic geodesics (which lie within a bounded distance from it in the graph metric, by graph-quasiconvexity).

Step 5 Conclude that the segments of β_{ea} that lie off *partially electrocuted horospheres* in fact lie outside a large ball about a fixed reference point if λ^h (the hyperbolic geodesic joining the end-points of λ in \tilde{S}^h does so. **Step**

6 Construct from $\beta_{ea} \subset \tilde{M}$ an electro-ambient quasigeodesic γ in \tilde{M}^h by

replacing pieces of β_{ea} that lie along partially electrocuted horoballs (if any) by hyperbolic quasigeodesics that lie (apart from bounded length segments at the beginning and end) within horoballs.

Step 7 Conclude that if λ^h lies outside large balls in S^h then each point of the path γ also lies outside large balls.

Step 8 Let γ^h be the hyperbolic geodesic joining the end-points of γ . Since the underlying set of γ^h lies in a neighborhood of γ , by Lemma 3.5, it must lie outside large balls. Hence by Lemma 2.1 there exists a Cannon-Thurston map and the limit set is locally connected.

Step 8 As in [29], the Steps 1-7 above are carried out first for manifolds of *p-incompressible boundary*. Then in the last step (as in Section 5.4 of [29]) the hypothesis is relaxed and the result proven for pared manifolds with incompressible boundary. (Recall from [29] that p-incompressibility roughly means the absence of accidental parabolics in any hyperbolic structure.) Note also that the definition of pared manifolds with incompressible boundary coincides with the notion of ‘incompressibility away from cusps’ introduced by Brock, Canary and Minsky in [8].

11 The Minsky Model and Split Geometry: A Sketch

The aim of this section is to sketch a proof of the following theorem:

Theorem [31]: Let M be a hyperbolic manifold corresponding to a totally degenerate surface group. Then M has split geometry.

We shall use a model manifold that was built by Minsky in [25] to prove the Ending Lamination Conjecture. It was shown by Brock, Canary and Minsky in [8] that the model is bi-Lipschitz equivalent to a hyperbolic manifold with the same ending laminations.

We refer the reader to Minsky [25] for the definitions of the relevant terms, particularly **hierarchy**, **resolution** and other related notions. Fix a hyperbolic surface S .

Step 1: Constructing a sequence of split surfaces

We require the following:

1. **Resolution sweep** - Lemma 5.8 of [25]

2. **J(v) is an interval** - Lemma 5.16 of [25]. What this means is the following:
 Given a vertex v (corresponding to a simple closed curve on the surface) occurring in the hierarchy H obtained from the ending laminations, fix a resolution $\{\tau_i\}_{i \in \mathcal{I}}$ of H with \mathcal{I} a subinterval of \mathbb{Z} . In the doubly (resp. simply) degenerate case \mathcal{I} can be thought of as \mathbb{Z} (resp. \mathbb{N}). Let

$$J(v) = \{i \in \mathcal{I} : v \in \mathbf{base} \mu_{\tau_i}\}$$

where $\mathbf{base} \mu_{\tau_i}$ denotes the pants decomposition induced by the marking μ_{τ_i} . We might as well assume that there are no repetitions in $J(v)$ (see the proof of Theorem 8.1 in [25]). Then $J(v)$ is an interval.

3. Again from the proof of Theorem 8.1 of [25] we obtain a flat orientation preserving embedding of the Minsky model minus Margulis tubes (denoted as $M_\nu(0)$) into $S \times \mathbb{R}$.
4. To each τ_i Minsky associates a *split-level surface* F_i . **This is the point at which the notions we have introduced in this paper and its predecessor [30] converge with those in Minsky's construction of his model in [25].**

In fact, the term *split geometry* was chosen with this in view. In what follows, we shall construct *split surfaces* (as per our definitions) from the *split level surfaces* of Minsky.

From the Minsky model we shall construct:

1. A sequence of split surfaces S_i^s exiting the end(s) of M . These will determine the levels for the split blocks and split geometry. There is a lower bound on the distance between S_i^s and S_{i+1}^s
2. A collection of Margulis tubes \mathcal{T} .

3. For each complementary annulus of S_i^s with core σ , there is a Margulis tube T whose core is freely homotopic to σ and such that T intersects the level i . (What this roughly means is that there is a T that contains the complementary annulus.)
4. For all i , either there exists a Margulis tube splitting both S_i^s and S_{i+1}^s and hence B_i^s , or else B_i is a thick block.
5. $T \cap S_i^s$ is either empty or consists of a pair of boundary components of S_i^s that are parallel in S_i .
6. There is a uniform upper bound n on the number of surfaces that T splits.

We define S_i^s to be the first split level surface in which v_i occurs. The region between S_i^s and S_{i+1}^s is temporarily designated B_i^s . We shall describe in [31] a procedure for interpolating a uniformly bounded number of split surfaces between S_i^s and S_{i+1}^s .

It will be shown in [31] that there exists n such that each thin Margulis tube splits at most n split surfaces in the above sequence.

This allows us to conclude that the Minsky model has weak split geometry.

Step 2: Graph quasiconvexity of Split Components

In order to prove that the Minsky model enjoys the property of **split geometry**, we need to show further that any of the split components is (not necessarily uniformly) quasiconvex in the hyperbolic metric, and uniformly quasiconvex in the graph metric, i.e. we require to show *hyperbolic quasiconvexity* and *uniform graph quasiconvexity* of split components.

Step 2A: Hyperbolic quasiconvexity is easy to prove and follows from the Thurston-Canary covering Theorem [35] [9].

Step 2B: Next, we need to prove that each split component of S_i^s corresponding to some subsurface Σ of S is uniformly graph quasiconvex. First off, any simple closed curve in Σ must be realised within a uniformly bounded distance in the graph metric. To prove this, we show in [31] that any pleated surface which contains at least one boundary geodesic of Σ in its pleating

locus is realised within a uniformly bounded distance of S_i^s in the graph metric.

Next, any split component is bounded by Margulis tubes. We drill out these tubes and appeal to the Drilling Theorem [7] to conclude that the drilled manifold and the complement of the Margulis tube in the original manifold are both uniformly bi-Lipschitz to the corresponding hyperbolic manifolds. Now in the drilled manifold the subsurface Σ gives us a genuine quasifuchsian group, whose convex hull boundary is pleated and hence within a uniform distance in the graph metric from the split component.

But the convex hull CH_Σ of a lift $\tilde{\Sigma}$ in the drilled hyperbolic manifold may also be regarded as a quasiconvex set in the hyperbolic manifold corresponding to the surface group. (This requires some additional argument which is supplied in [31].)

Since (using this identification) CH_Σ is uniformly graph quasiconvex in the drilled manifold, it is also uniformly graph quasiconvex in the split geometry model for \tilde{M} .

This shows that the Minsky model is of split geometry. Combining this fact with Theorems 9.2 and 9.3 we shall obtain:

Theorem: [31] Let ρ be a representation of a surface group H (corresponding to the surface S) into $PSL_2(C)$ without accidental parabolics. Let M denote the (convex core of) $\mathbb{H}^3/\rho(H)$. Further suppose that $i : S \rightarrow M$, taking parabolic to parabolics, induces a homotopy equivalence. Then the inclusion $\tilde{i} : \tilde{S} \rightarrow \tilde{M}$ extends continuously to a map $\hat{i} : \hat{S} \rightarrow \hat{M}$. Hence the limit set of \tilde{S} is locally connected.

12 Extending the Sullivan-McMullen Dictionary

A celebrated theorem of Yoccoz in Complex Dynamics (see Hubbard [18], or Milnor [22]) proves the local connectivity of certain Julia sets using a technique called ‘puzzle pieces’. We shall not describe this in any detail. What we shall simply say is that it consists of a decomposition of a complex domain into pieces each of which under iteration by a quadratic map converges to a single point. The dynamical system can then be regarded as a semigroup \mathbb{Z}_+ of transformations acting on a complex domain.

In the case of split (or amalgamation) geometry each of the split (or amalgamation) components can be regarded as a 3-dimensional analogue of puzzle pieces. Let us try to justify this analogy. Suppose there is a group G acting on the manifold \tilde{M} . Let $H \subset G$ denote the fundamental group of

a split component. Let G/H denote the coset space. Then what we require first is that if one takes a sequence of elements g_i going to infinity in the coset space, the iterates of the split component converge to a point in the limit sphere. However, this does not give all the information as G does not act co-compactly on \tilde{M} . In the cases we are interested in G/H correspond to normal directions to the split component lying within the block containing the split component. This does not help. To compensate, we look at the graph model. Here, there is no group in sight. However, normal directions can be salvaged from the **graph metric**. Thus, instead of going to infinity by iteration, we go to infinity in the graph metric. Further, the analogue of the requirement that iterates go to infinity, is that the visual diameter goes to zero as we move to infinity in the graph metric. This is easily ensured by hyperbolic quasiconvexity, and also follows easily from **graph quasiconvexity**. Note that **graph quasiconvexity** is a statement that gives uniform shrinking of visual diameter to zero as one goes to infinity.

Thus we extend the Sullivan-McMullen dictionary (see [34], [20]) between Kleinian groups and complex dynamics by suggesting the following analogy:

1. *Puzzle pieces* are analogous to **split components**
2. *Convergence to a point under iteration* is analogous to **graph quasiconvexity**

One issue that gets clarified by the above analogy is a point raised by McMullen in [21]. McMullen indicates that though the Julia set $J(P_\theta)$, where

$$P_\theta(z) = e^{2\pi i\theta}z + z^2$$

need not be locally connected in general by a result of Sullivan [33], the limit set of the punctured torus groups are nevertheless locally connected. By extending the analogy of puzzle pieces, this issue is to an extent clarified.

An analogue of the \mathbb{Z}_+ dynamical system may also be extracted from the split geometry model. Note that each block corresponds to a splitting of the surface group, and hence an action on a tree. As $i \rightarrow \infty$, the split blocks B_i^s and hence the induced splittings also go to infinity, converging to a **free action of the surface group on an \mathbb{R} -tree dual to the ending**

lamination. Thus iteration of the quadratic function corresponds to taking a sequence of splittings of the surface group converging to a (particular) action on an \mathbb{R} -tree.

Problem: The building of the Minsky model and its bi-Lipschitz equivalence to a hyperbolic manifold [25] [8] gives rise to a speculation that there should be a purely combinatorial way of doing much of the work. Bowditch's rendering [5], [6] of the Minsky, Brock-Canary-Minsky results is a step in this direction. This paper brings out the possibility that the whole thing should be do-able purely in terms of actions on trees. Of course there is an action of the surface group on a tree dual to a pants decomposition. So we do have a starting point. However, one ought to be able to give a purely combinatorial description, *ab initio*, in terms of a sequence of actions of surface groups on trees converging to an action on an \mathbb{R} -tree. This would open up the possibility of extending these results (including those of this paper) to other hyperbolic groups with infinite automorphism groups, notably free groups.

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